Improving efficiency of finite plans by optimal choice of input sets

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Abstract. Finite plans proved to be an efficient method to steer complex control systems via feedback quantization. Such finite plans can be encoded by finite–length words constructed on suitable alphabets, thus permitting transmission on limited capacity channels. In particular flat systems can be steered computing arbitrarily close approximations of a desired equilibrium in polynomial time.
The paper investigates how the efficiency of planning is affected by the choice of inputs, and provides some results as to optimal performance in terms of accuracy and range. Efficiency is here measured in terms of computational complexity and description length (in number of bits) of finite plans.

1 Introduction

Consider the problem of planning inputs to efficiently steer a controllable dynamical system of the type

$$\dot{x} = f(x, u),\ x \in X \subseteq \mathbb{R}^n, u \in U \subset \mathbb{R}^r \tag{1}$$

between neighborhoods of given initial and final equilibria. By any approximation procedure, one may achieve finite plans (for specific choice of initial and finite states). However, we aim at designing finite plans, among equilibria of the system, with short description length (measured in bits) and low computational complexity.

Concerns about the complexity of describing plans show up whenever communication or storage limitations are in place. Particularly fitting to this perspective are examples from robotics, where input symbols may represent commands (aka *behaviors*, or *modes*.) For instance, for autonomous mobile rovers, high level plans may be comprised of sequences of motion primitives such as `wander`, `look_for`, `avoid_wall`, etc.; in the control of humanoids (see e.g. [17]), symbols are encountered such as `walk`, `run`, `stop`, `squat`, etc.. To deal with real implementations, such languages must be able to encode the richest variety of tasks by words of the shortest length. Consider for instance the case where the robotic agent receives its reference plans from a remote high-level control center through a finite capacity communication channel, or plans are exchanged in a networked system of a large number of simple semi-autonomous agents. In general, it can be assumed that robots are capable of accepting finitely-described reference signals, and can implement a finite number of possible different feedback strategies via the use of embedded controllers, according to the received messages.

Finite plans steering was considered by many authors in recent years, e.g. [15, 9, 12]. A general framework was proposed by introducing Motion Description Languages in [4]. The line of research addressing finite hyerarchic abstractions of continuous systems via bi-simulations ([21, 22, 20]) has several contact points with the one presented in this paper. Of direct relevance to work presented here is the quantitative analysis of the specification complexity of input sequences for a class of automata, presented in [10]. The key result there is that feedback can substantially reduce the specification complexity (i.e., the description length of the shortest admissible plan) to reach a certain goal state.

In this paper we treat the more complex case of controlled dynamical systems and, by introducing *control encoding* of a symbolic input language, we can compute in polynomial time plans for flat systems, whose specification complexity is logarithmic in the size of the region to be covered. In our context, we postulate that control decoders are available and embedded on the remotely controlled plant. Decoders receive symbols from the planner, and translate them in suitable control actions, possibly based on locally available state information.

The result is obtained following this reasoning. First we seek for a symbolic encoding so that there exists a sublanguage, whose action on the system has the desirable properties of additive groups, i.e. the actions of control words are invertible and commute. Furthermore, under the action of words in this language, the reachable set becomes a lattice. More precisely, a suitable (dynamic) feedback encoding permits us to transform any flat system to:

$$z^+ = z + \bar{H}\mu, \quad \bar{H} \in \mathbb{R}^{n\times n},\ \mu \in \mathbb{Z}^n. \tag{2}$$

Once reduced to this special form, we address the problem of optimally choosing finite input sets in order to optimize the efficiency of plans. This objective is achieved by the study on the minimal specification complexity for interval-filling controls, derived from concurrent work of number-theoretic nature.
The effectiveness of the method is illustrate by Proposition 4.

1.1 Problem Description

Assume that system (1) is completely controllable, i.e. for any given two points x_0, x_f, a *plan* (i.e., a finite-support input function $u : [t, T + t] \to U$) exists that steers (1) from x_0 to x_f. An exact plan among initial and final point would generically require an infinite–length description, thus we consider approximate steering and address the following question:

Problem *Π*: Given a compact subset $\mathcal{M} \subseteq X$ and a tolerance ε, provide a specification P of plans such that, for every pair $(x_0, x_f) \in \mathcal{M}^2$, it exists a plan in P steering the system (1) from x_0 to within an ε-neighborhood of x_f.

We look for an *efficient* solution to this problem, where efficiency is intended in terms of low computational complexity, i.e. minimal number of elementary computations to be executed, and in terms of low specification complexity, i.e. minimal number of bits necessary to represent the plan (cf. [10]).

2 Encoding control quanta

Symbolic control is inherently related to the definition of elementary control events, or atoms, or *quanta*:

Definition 1. *A* control quantum *is a couple* (u, T) *where* $u : X \to L^\infty(\mathbb{R}^+ \times X, U)$ *and* $T : X \to \mathbb{R}^+$. *The set of control quanta is denoted by* $\widetilde{\mathcal{U}}$.

Hence, a control quantum is essentially a feedback that is applied to the system, starting at point x_0 at time t_0, until time $t_0 + T(x_0)$. To each control quantum it is natural to associate the map $\phi_{(u,T)} : X \to X$, where $\phi_{(u,T)}(x_0)$ is the solution at time $T(x_0)$ of the Cauchy problem corresponding to initial data x_0 and control $u(x_0)$.

Definition 2. *A* control quantization *consists in assigning a finite set* $\mathcal{U} \subset \widetilde{\mathcal{U}}$. *A* (symbolic) control encoding *on a control quantization is a map* $E : \Sigma \to \mathcal{U}$, *where* $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ *is a finite set of symbols.*

Given a control quantization and an encoder, we have the diagram $\Sigma \xrightarrow{E} \mathcal{U} \xrightarrow{\phi} \mathcal{D}(X)$, where $\mathcal{D}(X)$ denotes the group of automorphisms on X. This can be extended in an obvious way to $\Sigma^* \xrightarrow{E^*} \mathcal{U}^* \xrightarrow{\phi^*} \mathcal{D}(X)$, where Σ^* is the set of words formed with letters from the alphabet Σ, including the empty string ϵ. We assume $\phi \circ E(\epsilon) = Id(X)$, i.e. the identity map in $\mathcal{D}(X)$. An action of the monoid Σ^* on X is thus defined. In general, being the action of Σ^* just a monoid, the analysis of its action on the state space can be quite hard, and the structure of the reachable set under generic quantized controls can be very intricated (even for linear systems: see e.g. [1, 6, 2]). However, we will show that, appropriately choosing the quantization, for every flat system it is possible to find a sub-language Ω of Σ^* acting on $\mathbb{R}^n$ as $\mathbb{Z}^n$. Therefore, in suitable state and input coordinates, the system takes the form (2).
To reach the desired special form (2), we focus our attention on designing encodings that achieve simple composition rules for the action of words in a sub-language $\Omega \subset \Sigma^*$:

$$\forall \omega \in \Omega, \exists h(\omega) \in \mathbb{R}^n : \forall x \in X, (\phi^* \circ E^*(\omega))(x) = x + h(\omega), \tag{3}$$

and

$$\forall \omega_1 \in \Omega, \exists \bar{\omega}_1 \in \Omega : (\phi^* \circ E^*(\omega_1)) \circ (\phi^* \circ E^*(\bar{\omega}_1)) = Id(X). \tag{4}$$

The additivity rule (3) implies that actions commute, therefore, the global action is independent from the order of application of control words in Ω. Moreover we have the following:

Proposition 1. *Under rules (3), (4), there exists a sublanguage $\Omega' \subset \Omega$ such that the corresponding reachable sets are lattices.*

Proof. First notice that, by rules (3) and (4), Ω acts on the states as an additive group. As a consequence, the reachable set from any point in X under the concatenation of words in Ω is a set Λ generated by vectors $h(\omega), \omega \in \Omega$,

$$\Lambda = \{h(\omega_1)\lambda_1 + \cdots + h(\omega_N)\lambda_N | \lambda_i \in \mathbb{Z}, N \in \mathbb{N}\}.$$

If $h(\omega) \in \mathbb{Q}^n$, $\forall \omega \in \Omega$, then we can choose $\Omega' = \Omega$. Otherwise, we choose Ω' to consist of concatenations of only n words in Σ^* which produce independent vectors $h(\omega)$.

A further important concern is that system (1) under symbolic control, maintains the possibility of approximating arbitrarily well all reachable equilibria in its state space, for suitable choices of symbols.

Definition 3. *A control system $\dot{x} = f(x, u)$ is* additively *(or* lattice*)* approachable *if, for every $\varepsilon > 0$, there exist a control quantization $\mathcal{U}_\varepsilon$ and an encoding $E^* : \Omega \mapsto \mathcal{U}_\varepsilon^*$ with $card(\mathcal{U}_\varepsilon) = q \in \mathbb{N}$, such that: i) the action of Ω obeys (3), (4), and ii) for every $x_0, x_f \in X$, there exists x in the Ω–orbit of x_0 with $\|x - x_f\| < \varepsilon$.*

Remark 1. The reachable set being a lattice under quantization does not imply additive approachability. For instance, consider the example used in [14] to illustrate the so–called kinodynamic planning method.This consists of a double integrator $\ddot{q} = u$ with piecewise constant encoding $\mathcal{U} = \{u_0 = 0, u_1 = 1, u_2 = -1\}$ on intervals of fixed length $T = 1$. The sampled system reads

$$q^+ = q + \dot{q} + \frac{u}{2}, \qquad \dot{q}^+ = \dot{q} + u,$$

hence $q(N) = q(0) + N\dot{q}(0) + \sum_{i=1}^{N} \frac{2(N-i)+1}{2} u(i)$, $\dot{q}(N) = \dot{q}(0) + \sum_{i=1}^{N} u(i)$. The reachable set from $q(0) = \dot{q}(0) = 0$ is

$$R(\mathcal{U}, 0) = \left\{ \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \lambda,\ \lambda \in \mathbb{Z}^2 \right\}.$$

The quantization thus induces a lattice structure on the reachable set. The lattice mesh can be reduced to any desired ε resolution by scaling U or T. However, the actions of control quanta do not compose according to rule (3): indeed, $\phi^*(u_1 u_2) \neq \phi^*(u_2 u_1)$ (for instance, $\phi^*(u_1 u_2)(0, 0) = (1, 0)$, while $\phi^*(u_2 u_1)(0, 0) = (-1, 0)$).

The following theorem motivates the interest in seeking control encodings for additive approachability, moreover Theorem 3 below shows the applicability of the method.

Theorem 1. *For an additively approachable system, a specification P for problem Π can be given in polynomial time.*

Proof. Consider a feedback encoding ensuring additive approachability. Arrange a sufficient number q of action vectors $h(\omega_i)$, $\omega_i \in \Omega$ in the columns of a matrix $H \in \mathbb{R}^{n\times q}$. The reachable set from x_0 is thus a lattice $x_0 + \Lambda$, where $\Lambda = \{H\lambda | \lambda \in \mathbb{Z}^q\}$. Additive approachability guarantees that the dispersion of Λ can be bounded by $\frac{1}{2}\varepsilon$, hence, $\forall x_f$, $\exists y \in \Lambda : \|x_f - x_0 - y\| \leq \varepsilon$. Finding a plan to x_f is thus reduced to solving the system of diophantine equations

$$y = H\lambda. \tag{5}$$

Each lattice coordinate λ_i represent directly the number of times the control word ω_i, hence the corresponding sequence of control quanta, is to be used to reach the goal. Due to additivity of the action, the order of application of the ω_i is ininfluent. The linear integer programming problem (5) can be solved in polynomial time with respect to the state space dimension n and p. Indeed, write H in Hermite normal form, $H = [L\ 0]\,U$, where $L \in \mathbb{R}^{n\times n}$ is a nonnegative, lower triangular, nonsingular matrix, and $U \in \mathbb{Q}^{m\times m}$ is unimodular (i.e., obtained from the identity matrix through elementary column operations). Once the Hermite normal form of H has been computed (which can be done off-line in polynomial time [18, 23]), all possible plans to reach any desired configuration y are easily obtained as $\lambda = U^{-1}[L^{-1}y, \mu]$, $\forall \mu \in \mathbb{Z}^{m-n}$.

2.1 Reducing the specification complexity

We now address the specification complexity for problem Π for a system in form (2). Without loss of generality to the purposes of this section, we can set the tolerance $\varepsilon = 1$ and assume $\bar{H} = Id$, thus reducing to system

$$z^+ = z + u. \tag{6}$$

This system can be treated componentwise, hence it will be sufficient to consider (6) with $z \in \mathbb{R}$. To deal with problem Π we introduce the following problems. Consider system (6) and fix integers $m > 0$, $N > 0$ and $M > 0$. Our aim is to study, for every integer control set $\mathcal{W} = \{0, \pm v_1, \ldots, \pm v_m\}$, the reachable set $R(0, N)$ from the origin in N steps . More precisely we want to determine the maximal M such that the interval of integers $I(M) = [-M, -M+1, \ldots, M] \subset \mathbb{Z}$ is contained in $R(0, N)$.

We can thus state three significant problems:

Problem 1. Given a fixed number m and a symmetric interval of integers $I(M)$, find the minimal number N of steps and the set of $2m+1$ control values to completely fill $I(M)$ in at most N steps.

Problem 2. Given a fixed number N of steps and a symmetric interval of integers $I(M)$, find the minimal m such that there exists a control set with $2m+1$ elements which completely fills $I(M)$ in at most N steps.

Problem 3. Given a fixed number m and N of steps, find the optimal choice of $2m+1$ control values to completely fill a maximal symmetric interval of integers $I(M)$ in at most N step.

Notice that each problem is obtained fixing two of the three parameters m, N and M and optimizing over the other two. To treat Problem Π, Problems 1 and 2 are relevant: in both cases M is fixed and the optimization reduces the specification complexity. However, it is exactly Problem 3 which is mostly treatable. Thus we now focus on Problem 3 and, later, derive some information on Problems 1 and 2 from the solution of Problem 3.

Problem 3 is a number theoretical problem, related but not equivalent to the well-known "Frobenius postage stamp problem". More precisely, the postage problem seeks to maximize the minimum postage fee not realizable using stamps from a finite set of m possible denominations. For the classical postage problem, only results for small values of m are known, see [13]. The main difference with Problem 3 is the positivity of stamp denominations, while control values from $\mathcal{W}$ are also negative. Although this difference has substantial technical implications, the difficulty of the two problems is comparable.

Problem 3 was first studied in [5], then solved for $m = 2, 3, 4$ and any N in [7], where a general asymptotic formula was conjectured for every m. We report here the explicit formulae for the optimal choice of controls for $m = 2, 3$. For $m = 2$ we simply obtain $v_1 = N$ and $v_2 = N + 1$. For $m = 3$ we get:

$$v_3 = \begin{cases} N^2/4 + 3/2\,N + 5/4 & \text{if } N \text{ is odd} \\ N^2/4 + 3/2\,N + 1 & \text{if } N \text{ is even,} \end{cases}$$

$$v_2 = v_3 - 1,$$

$$v_1 = \begin{cases} v_3 - \frac{N+1}{2} - 1 & \text{if } N \text{ is odd} \\ v_3 - \frac{N}{2} - 2 & \text{if } N \text{ is even.} \end{cases}$$

Table 1 reports the maximum interval of the horizontal line which can be covered with unit resolution and different word lengths N, along with the actual values of the different control sets, for $m = 3$ and $m = 4$.

For $m = 2, 3, 4$ and $N >> m$, for the largest value in $\mathcal{W}$ it holds asymptotically $v_m \sim (\frac{N}{m-1})^{(m-1)}$. Given $2m+1$ controls one can thus reach in N steps a region of size

$$M \sim N^m / m^m. \tag{7}$$

In [7], it is conjectured that (7) holds for every m.

Consider now again Problem 1. In this case m and M are fixed. From (7), we know that we can cover $I(M)$, taking the $2m+1$ optimal control values for Problem 3, in

$$N \sim m\, M^{\frac{1}{m}} \tag{8}$$

N	1	2	3	4	5	6	7
v_1	1	3	5	8	11	15	19
v_2	2	4	7	10	14	18	23
v_3	3	5	8	11	15	19	24
M	3	10	24	44	75	114	168

N	1	2	3	4	5	6	7
v_1	1	3	7	13	19	29	41
v_2	2	6	9	18	27	36	52
v_3	3	7	11	20	29	39	55
v_4	4	8	12	21	30	40	56
M	4	16	36	84	150	240	392

Table 1. Optimal interval-filling input values for system (6) for $m = 3$ (above) and $m = 4$ (below).

steps. This gives an approximate solution to Problem 1.
On the other hand, for Problem 2 (now N and M are fixed), taking the $2m+1$ optimal control values for Problem 3, we can cover $I(M)$ in N steps using $2m+1$ controls where

$$m \sim \frac{N}{M^{\frac{1}{m}}}. \tag{9}$$

Again this gives an approximate solution to Problem 2.
To efficiently solve Problem *Π* we need to reduce the specification complexity of finite plans. In order to achieve that we may either use the solution to Problem 1 or to Problem 2 In both cases, to describe plans covering the region of size M, a sequence of length N of symbols from an alphabet of size $2m+1$ should be given. This results on a specification complexity of $N\lceil\log_2(2m+1)\rceil$. Therefore we immediately get the following:

Proposition 2. *Using the (approximate) solutions to Problems 1 and 2, we can cover the region $I(M)$ by finite plans with specification complexities asymptotically given by (respectively):*

$$m\,M^{\frac{1}{m}}\lceil\log_2(2m+1)\rceil\,, \tag{10}$$

$$N\left\lceil\log_2\left(\frac{2N}{M^{\frac{1}{m}}}+1\right)\right\rceil \tag{11}$$

Clearly the two expressions (10) and (11) have the same asymptotic behavior (for $M\to\infty$), thus we focus on the first which depends only on two parameters m and M.
One can check, by formal computations, that (10) admits a minimum in m. We report in figure 1 the graph of (10) for $M = 10^3$: note the discontinuities produced by the function $\lceil\cdot\rceil$. An exact expression for the minimum is not possible,

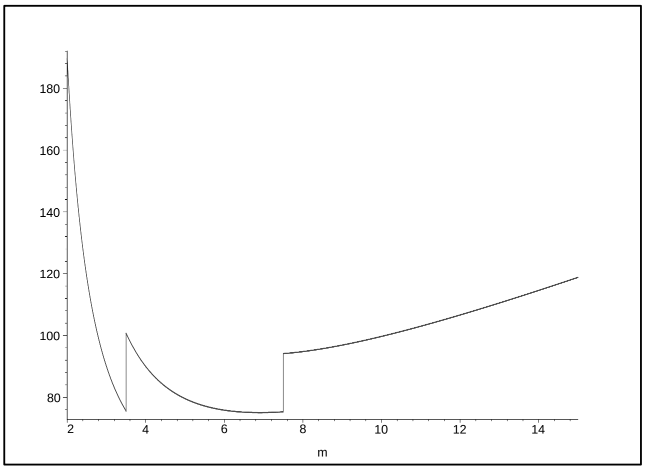

Fig. 1. Graph of (10) for $M = 1000$.

however we can compute the derivative of (10) (replacing the function $\lceil\cdot\rceil$ with the identity) and thus obtaining:

$$\frac{M^{\frac{1}{m}}}{\ln(2)}\left(\ln(2m+1)\left(1-\frac{\ln(M)}{m}\right)+\frac{2m}{2m+1}\right).$$

From this expression, we see that the optimal value m^* satisfies $m^* < \ln(M)$. Finally, replacing this value in (10) we obtain:

Proposition 3. *Using the (approximate) solutions to Problems 1, the specification complexity asymptotically satisfies:*

$$\mathcal{C} \leq \ln(M)\, M^{\frac{1}{\ln(M)}} \left\lceil \log_2(2\,\ln(M)+1)\right\rceil,$$

A compact representation of control sequences is obtained by using Run–Length Encoding (RLE). RLE consists in replacing repeated runs of a single symbol in an input stream by a single instance of the symbol and a run count. This compression method is particularly well suited for our method, because of the commutativity of symbols in control strings. In fact, we can assign, for each possible control value, an integer of size at most N, specifying how many times the corresponding control must be used. In this way, the control sequence requires $(2m+1)\lceil\log_2(N)\rceil$ bits, or rather, by exploiting the symmetry of the symbol set and using sign-magnitude representation, $(m+1)\,(1+\lceil\log_2(N+1)\rceil)$ bits. (We are assuming that control values are already computed off-line.) Using Proposition 3 and again (7), we thus get:

Proposition 4. *For Problem Π, using feedback encoding, the approximate solution to Problem 1 and RLE, the specification complexity $\mathcal{C}$ satisfies:*

$$\mathcal{C} \sim (m+1)\left(1+\left\lceil\log_2(mM^{\frac{1}{m}}+1)\right\rceil\right) \qquad (12)$$

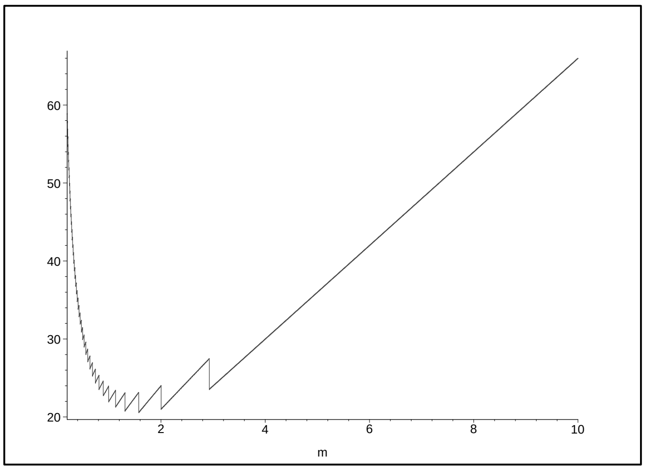

Fig. 2. Graph of (12) for $M = 1000$.

We can study this expression as above to determine an optimal value m^* of m: see in figure 2 the graph of (12) for $M = 10^3$. However, in this case we can only estimate $m^* = o(\ln(M))$, thus

$$\mathcal{C} = o(\ln(M)) \log_2(o(\ln(M))M)$$

3 Feedback Encoding for flat systems

Feedback encoding consists in associating to each symbol a control input u that depends on the symbol itself, on the current state of the system, and on its structure. If the encoding incorporates memory elements, e.g. additional states ξ are used to define the feedback, the feedback encoding is referred to as dynamic. The method of feedback encoding avails symbolic control with powerful results from the literature on feedback equivalence of dynamical systems. We show how this can be exploited to apply the planning method of theorem 1 to the rather general class of flat systems.

We start treating the case of linear systems:

$$\dot{x} = Fx + Gu \tag{13}$$

with $x \in \mathbb{R}^n$, $u \in U = \mathbb{R}^r$ and rank $G = r$. Application to (13) of piecewise constant encoding of symbolic inputs with durations $T_i = T$, $\forall i$, generates the discrete-time linear system

$$x^+ = Ax + Bu, \tag{14}$$

with $A = e^{FT}$, $B = (\int_0^T e^{(T-s)F} ds)G$. Let us recall the definition of Brunovsky form (see e.g. [19]). For a controllable system (14), there exist a change of coordinates S in the state space and V in the input space, and a linear feedback matrix K_0 such that the new system with drift $\tilde{A} = S^{-1}(A+BK_0)S$ and control matrix $\tilde{B} = S^{-1}BV$ has the following properties. The state $\xi = S^{-1}x$ can be split in r subvectors $\xi = (\xi_1, \ldots, \xi_r)$ for which the dynamics are written as

$$\dot{\xi}_i = A_{\kappa_i}\xi_i + b_{\kappa_i}v'_i, \; i = 1, \ldots, r \tag{15}$$

where $\xi_i \in \mathbb{R}^{\kappa_i}$,

$$A_{\kappa_i} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\kappa_i \times \kappa_i}, \qquad b_{\kappa_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\kappa_i},$$

$v'_i \in \mathbb{R}$ and $\sum_{i=1}^{r} \kappa_i = n$.

Theorem 2. *For a controllable linear discrete-time system $x^+ = Ax+Bu$, there exists an integer $\ell > 1$ and a linear feedback encoding $E : \sigma_i \mapsto Kx + w_i$ with constant $K \in \mathbb{R}^{n\times n}$ and $w_i \in \mathcal{W}$, $\mathcal{W} \subset \mathbb{R}^r$ a quantized control set, such that, for all subsequences of period ℓT extracted from $x(\cdot)$, the reachable set is a lattice of arbitrarily fine mesh. In other words the all controllable linear discrete-time systems are additively approachable.*

We recall preliminarily a result which can be derived directly from [2].

Lemma 1. *The reachable set of the scalar discrete time linear system $\xi^+ = \xi+v$, $\xi \in \mathbb{R}$, $v \in \mathcal{W}:=\gamma W$ with $\gamma > 0$ and $W = \{0, \pm w_1, \ldots, \pm w_m\}$, $w_i \in \mathbb{N}$ with at least two elements w_i w_j coprime, is a lattice of mesh size γ.*

Proof. Theorem 2.
For the controllable pair (A, B), let S, V, and K_0 be matrices such that $(S^{-1}(A+BK_0)S,\ S^{-1}BV)$ is in Brunovsky form. Let $v' = K_1\xi + v$, where:

- $v \in \mathcal{W} = \gamma_1\,{}^1W \times \cdots \times \gamma_r\,{}^rW$, with ${}^kW = \{0, \pm{}^kw_1, \ldots, \pm{}^kw_{m_k}\}$, ${}^kw_j \in \mathbb{N}$ $k = 1, \ldots, r$, $j = 1, \ldots, m_k$, each ${}^kW$ including at least two coprime elements ${}^kw_i$ ${}^kw_j$;
- $K_1 \in \mathbb{R}^{r\times n}$ such that its i–th row (denoted K_{1i}) contains all zeroes except for the element in the $(\kappa_{i-1}+1)$–th column which is equal to one (recall that by definition $\kappa_0 = 0$).

Using notation as in (15), it can be easily observed that $(A_{\kappa_i} + B_{\kappa_i}K_{1i})^{\kappa_i} = I_{\kappa_i}$, the $\kappa_i \times \kappa_i$ identity matrix. Hence, if we let $\ell =$ l.c.m. $\{\kappa_i : i = 1, ..., r\}$, we get $\left[S^{-1}((A + BK_0)S + BVK_1)\right]^{\ell} = I_n$.
Let $\xi_i \in \mathbb{R}^{\kappa_i}$ denote the i–th component of the state vector relative to the pair $(A_{\kappa_i}, B_{\kappa_i})$. For any $\tau \in \mathbb{N}$ we have $\xi_i(\tau+\kappa_i) = \xi_i(\tau) + [v_i(\tau), \ldots, v_i(\tau+\kappa_i-1)]$. On the longer period of ℓT, we have

$$\begin{aligned} \xi_i(\tau+\ell) &= \xi_i(\tau) + \begin{bmatrix} \sum_{k=0}^{\frac{\ell}{\kappa_i}-1} v_i(\tau + k\kappa_i) \\ \vdots \\ \sum_{k=0}^{\frac{\ell}{\kappa_i}-1} v_i(\tau + \kappa_i - 1 + k\kappa_i) \end{bmatrix} \\ &:= \xi_i(\tau) + \bar{v}_i(\tau), \end{aligned}$$

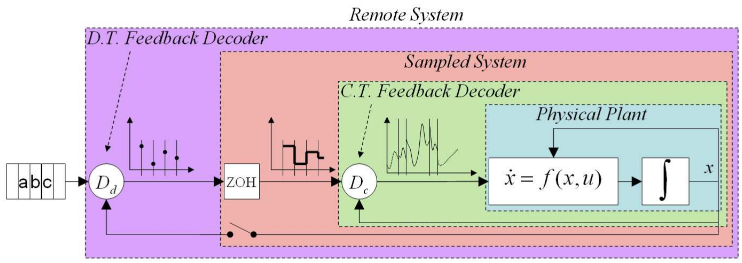

Fig. 3. Nested discrete-time continuous-time feedback encoding.

hence, in the initial coordinates,

$$x(\tau + \ell) = x(\tau) + S\bar{v}.$$

It is also clear that, for any ε, it is possible to choose Γ such that z can be driven in a finite number of steps (multiple of ℓ) to within an ε-neighborhood of any point in $\mathbb{R}^n$.

Let us now pass to treat a general system (1) and let the equilibrium set be $\mathcal{E} = \{x \in X | \exists u \in U, f(x,u) = 0\}$. The focus on equilibria is consistent with usual practice in control, where equilibrium configurations typically correspond to nominal working conditions for a system (possibly up to group symmetries, see e.g. [12]).
Among systems with drift, linear systems are the simplest, yet their analysis encompasses the key features and difficulties of planning. Indeed, our strategy to attack the general case consists of reducing to planning for linear systems via feedback encoding. To achieve this, we introduce a further generalized encoder (still encompassed by the above definition of control quanta), i.e. the *nested feedback encoding* described in fig. 3. In this case, an inner continuous (possibly dynamic) feedback loop and an outer discrete-time loop – both embedded on the remote system – are used to achieve richer encoding of transmitted symbols. Since additive approachability for linear systems is proved in theorem 2, using nested feedback encoding, all feedback linearizable systems are hence additively approachable. Recalling results from [11], we can state the following

Theorem 3. *Every differentially flat system is locally additively approachable.*

4 Example

We illustrate the power of the proposed method by solving the steering problem for an example in the class of underactuated mechanical systems, which have attracted wide attention in the recent literature (see e.g. [8]).
In particular, we consider the class of underactuated mechanisms identified as "$(n-1)X_a - R_u$ planar robots", i.e. mechanisms having $n-1$ active joints of any

type, and a passive rotational joint. In order to simplify the model analysis and control design, it is convenient to use a specific set of generalized coordinates. In particular, let $q = (q_1, ..., q_{n-3}, x, y, \theta) = (q_a, \theta)$ where (x, y) are the cartesian coordinates of the base of the last link. Assuming motion in a horizontal plane (or zero gravity), the dynamic model takes on the partitioned form

$$\begin{bmatrix} B_a(q_a) & \begin{matrix} 0_{(n-3)\times 1} \\ -m_n d_n s_\theta \\ m_n d_n c_\theta \end{matrix} \\ 0_{1\times(n-3)} \; -m_n d_n s_\theta \; m_n d_n c_\theta & I_n + m_n d_n^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{\theta} \end{bmatrix} + \\ + \begin{bmatrix} c_a(q, \dot{q}) \\ 0 \end{bmatrix} = \begin{bmatrix} F_a \\ 0 \end{bmatrix} \tag{16}$$

where $F_a = (F_1, ..., F_{n-3}, F_x, F_y)$ are the generalized forces performing work on the q_a coordinates, $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. For the n–th link, I_n, m_n and d_n are the baricentral inertia, the mass and the distance of the center of mass from its base.

In order to make the analysis independent from the nature of the $n-1$ active joints, the relative dynamics in (16) can be linearized via a globally defined partial static feedback, thus reducing them to a chain of two integrators per actuated joint. The dynamics of the coordinates q_i, $i = 1, ..., n-3$ are completely decoupled from the dynamics of the remaining coordinates (x, y, θ). Therefore, we will henceforth only consider the case $n = 3$. Following [8], we choose the cartesian coordinates of the center of percussion as the system's (flat) outputs:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + K_{CP} \begin{bmatrix} c_\theta \\ s_\theta \end{bmatrix}. \tag{17}$$

The dynamics of the system after the dynamic feedback linearization are written as $y_1^{(4)} = v_1$, $y_2^{(4)} = v_2$. Choosing a sample time $t = 1s$ we obtain the following discrete time linear system:

$$x_i^+ = Ax_i + Bv_i =$$

$$= \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} \frac{1}{24} \\ \frac{1}{6} \\ \frac{1}{2} \\ 1 \end{bmatrix} v_i$$

where $x_i = \left(y_i,\ y_i^{(1)},\ y_i^{(2)},\ y_i^{(3)} \right)$, $i = 1, 2$. Being each subsystem controllable, there exist S such that $(S^{-1}AS, S^{-1}B)$ is in control canonical form. For each subsystem in control canonical form, the set of equilibria is given by $\{\alpha\mathbf{1}_4 \in \mathbb{R}^4 : \alpha \in \mathbb{R}\}$. Then, in the initial coordinates, the set of equilibria is given by $\{\alpha S\mathbf{1}_4 \in \mathbb{R}^4 : \alpha \in \mathbb{R}\}$. For a given $\alpha \in \mathbb{R}$ we obtain the equilibrium $\alpha\mathbf{1}_4$

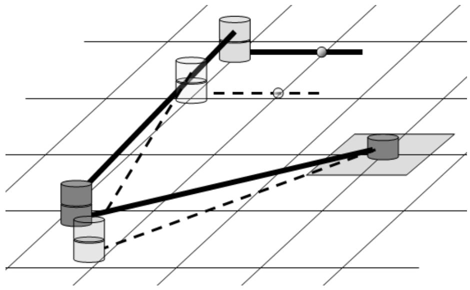

Fig. 4. An underactuated robot arm of type $2R_a - R_u$ used in example 2: the given initial and final configurations are shown by dashed and solid lines, respectively.

for the control canonical form and the equilibrium $(\alpha, 0, 0, 0)$ for the original subsystem, hence a constant position of the considered coordinate of the center of percussion. The scale factor is 1 in this case.

To obtain a reachable lattice of size $\gamma_1, \gamma_2 > 0$, ${}^1W$, ${}^2W$ can be chosen to be any finite sets of integers, such that at least two of its elements are coprime, and and inputs scaled as ${}^iv \in \gamma_i\, {}^iW$, $i = 1, 2$.

Given an initial robot pose $(y_1, y_2, \theta) = (0, 0, 0)$, consider three maneuvers: translation along the x axis, translation along the straight line $y = x$ and translation along the y axis. The first one can be achieved with a single symbol w applied on the input ${}^1v$ for $n = 4$ periods. The second maneuver is similar to the first one: we apply the previous command on the two inputs ${}^1v$ and ${}^2v$ for $n = 4$ periods. We can split the third maneuver in two maneuvers of the previous types.

Initial and final positions of a 3R robot are shown in fig. 4. Simulations were performed setting $l1 = l2 = 3m$, $K_{CP} = 1m$, $T = 1s$, and $w = 0.8\text{m/s}^2$. Fig. 5 shows the coordinates of the Center of Percussion of the last link while fig. 6 shows the angles of the active joints and the orientation of the last passive link, respectively.

5 Conclusions

In this paper, we addressed the issue of designing efficient finite plans to steer controlled dynamical systems. Efficiency is measured by specification and computational complexities.

Via suitable feedback encoding, based on control quanta, we showed how to reduce flat systems to a special form. Once this is obtained, we can use number-theoretic results to improve efficiency. It seems fair to affirm that few practically interesting classes of controllable systems remain outside the scope of application of the presented methods.

Connections to state observers in planning are unexplored at this stage.

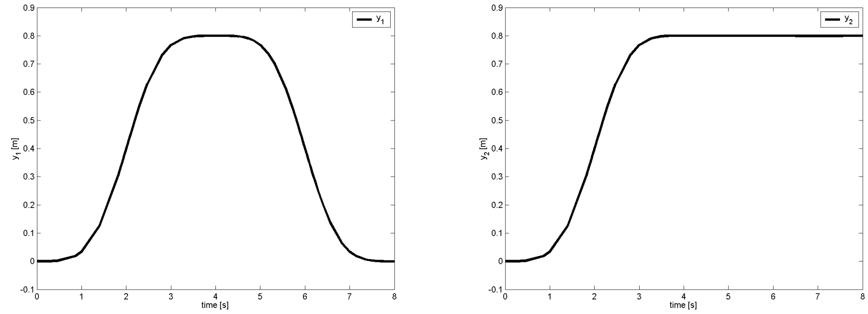

Fig. 5. Coordinates y_1 (left) and y_2 (right) of the center of percussion

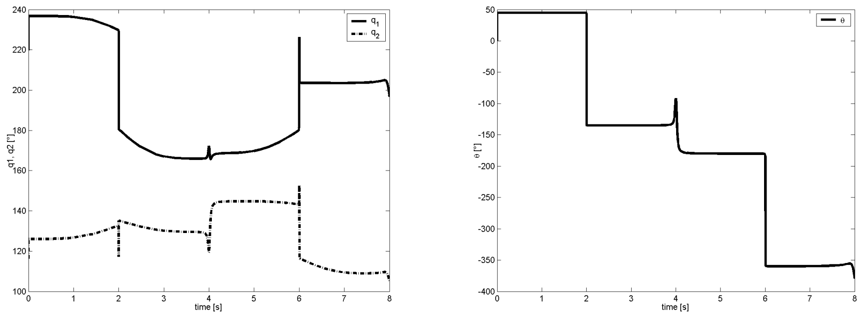

Fig. 6. Active joints angles (left) and orientation of the last passive link (right)

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