Encoding Steering Control with Symbols

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***Abstract*— In this paper, we consider the problem of steering complex dynamical systems among equilibria in their state space in efficient ways. Efficiency is considered as the possibility of compactly representing the (typically very large, or infinite) set of reachable equilibria and quickly computing plans to move among them. To this purpose, we consider the possibility of building lattice structures by purposefully introducing quantization of inputs. We consider different ways in which control actions can be encoded in a finite or numerable set of symbols, review different applications where symbolic encoding of control actions can be employed with success, and provide a unified framework in which to study the many different possible manifestations of the idea.**

I. Introduction

Although traditional system theory has been concerned separately with either discrete event systems or controlled dynamic systems with a continuous state space, the actual system configuration in many technical contexts involves a close interaction of such components (see fig. 1). Here, a physical plant, described in terms of a classical difference or differential controlled system, is interfaced to the controller via encoding of its outputs in a finite or countable set of symbols (say, an alphabet $L = \{A, B, C, \ldots\}$). On the other hand, control actions decided by the controller are also similarly encoded in a different alphabet $G = \{\alpha, \beta, \gamma, \ldots\}$ of symbols. Naturally, in this framework the feedback controller should be regarded as a (possibly dynamic) map form L to G, hence as an automaton. The cardinality of the I/O alphabets, and the dimension of the feedback automaton, are indicative of the complexity of the controller (cf. [1]). The most obvious occurrence of such a scheme is related to systems with quantization of inputs and outputs, a very common aspect of digital control, which has recently attracted renewed attention in the stabilization literature ([2], [3], [4], [5]).

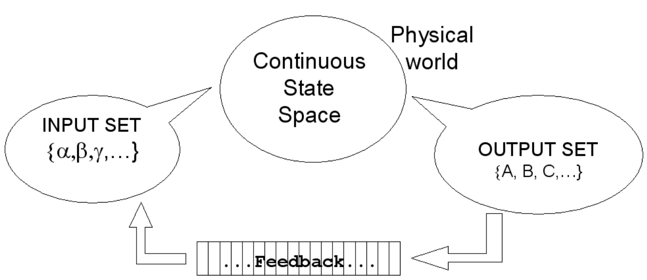

Fig. 1. Basic scheme of a hybrid analogic-symbolic feedback system

There are higher-level instances of the scheme of fig.1, where the physical plant represents a large system capable of complex behaviours, and logic control represents hierarchically abstracted levels of decision, planning and supervision. Particularly fitting to this perspective are examples from robotics, where input symbols may represent commands (aka *behaviors*, or *modes*), such as e.g. `walk, run, stop, squat,` etc. in the control of humanoids via linguistic primitives [6]. A framework for describing these systems, Motion Description Languages, has been introduced recently ([7], [8], [9]), while extensions to systems with symmetries have been presented in [10].

This paper is concerned with the problem of steering complex dynamical systems among equilibria in their state space. Indeed, we assume here that the system has a large (often infinite) set of equilibria, which are stable, possibly following the use of lower-level controllers we disregard here: typical is the case of driftless systems. Steering efficiency is considered as the possibility of compactly representing the set of reachable equilibria, and quickly computing plans to move among them. To this purpose, we consider the possibility of building suitable structures in the reachable set by purposefully introducing symbolic encoding of inputs. Indeed, previous work ([11], [12], [13], [14]) has shown that quantized (i.e., symbolic) inputs can induce particular structures on the reachable set of continuous sytems. In particular, while it may happen that a reachable system maintains the possibility of approaching arbitrarily close any point in the state space, under some circumstances the rechable set under quantized control is itself discrete, and possesses a lattice structure. Further work has shown that lattice structures can be turned to advantage in several applications such as, for instance, planning motions of complex systems ([15]) or applying efficient algorithms for optimal control computations ([16]).

In this paper we will study conditions under which the control of a continuous system can be encoded in symbols, so as to achieve some desirable properties related to the structure of its reachable set. To this purpose we will define a concept of "reticulability" of a system, meaning the existence of encoding schemes that make the reachable set a lattice, and of "approachability", meaning that the encoding can be tuned so as to make the lattice mesh arbitrarily fine. We will discuss conditions under which these properties are achievable. It so turns out that a hierarchy of encoders can be considered, from simpler schemes (such as piece-wise constant control signals) to more complex encoders (e.g. piecewise continuous). The hierarchy is topped by feedback encoders, which offer the richest possibility for achieving the desired structures. We will illustrate the results by several examples, ranging from linear systems to chained-form, nilpotent, and more general nonlinear systems.

II. BASIC DEFINITION

Consider a control system

$$\dot{x} = f(x, u) \tag{1}$$

where the state $x \in M$ evolves on a manifold, and let the control u take values in the continuous control space U. Symbolic control is inherently related to the definition of elementary control events, or atoms, or *quanta*. We will refer to the following

Definition 1: A *control quantum* is a couple (u, T) where $u : \mathbb{R}^+ \times M \to U$ and $T : M \to \mathbb{R}^+$. The set of control quanta is denoted by $\widetilde{\mathcal{U}}$.
In a natural way, we associate to each control quantum (u, T) a map $\phi_{(u,T)} : M \to M$: given $x_0 \in M$, $\phi_{(u,T)}(x_0)$ is the solution at time $T(x_0)$ of the Cauchy problem

$$\begin{cases} \dot{x} = f(x, u(t, x)) \\ x(0) = x_0. \end{cases} \tag{2}$$

If the time–varying vector field $f_u = f(x, u(t, x))$ is smooth with sublinear growth, and T is smooth, then the map $\phi_{(u,T)}$ is well defined and we write $\phi_{(u,T)}(x_0) = e^{T(x_0) f_u} x_0$. Under suitable conditions on T (e.g. T constant), denoting by $\mathcal{D}(M)$ the space of diffeomorphism of M onto itself, we have the map $\Phi : \widetilde{\mathcal{U}} \to \mathcal{D}(M),\ \Phi((u, T)) = \phi_{(u,T)}$.

Definition 2: A *control quantization* consists in assigning a finite or countable set $\mathcal{U} \subset \widetilde{\mathcal{U}}$. A *(symbolic) control encoder* on a control quantization is a map $E : \Sigma \to \mathcal{U}$, where $\Sigma = \{\alpha, \beta, \ldots\}$ is a finite or countable set of symbols.
In fig. 2, three different control encoding schemes are pictorially described. Piecewise constant encoding, where each control quantum $q_i = (u_i, T_i)$ has both u_i and T_i constant, is the simplest case. Piecewise smooth controls, where T_i is fixed, and u_i are smooth functions of time not depending from the state, allow for more powerful encoding - for instance, different u_i's may represent pieces of extremal controls to be pasted together in an approximate optimal control scheme. Finally, feedback encoding consists in associating to each symbol a control input u that depends on the symbol itself, on the current state of the system, and on its structure (e.g., the scheme can be regarded as generated by defining a feedback $u = f(x, r)$ for system (2), and a piecewise constant encoding on the reference r.)
Although the definition above is not the most general one could introduce (e.g., it could be generalized to couples (u, T) where $u : M \to L^\infty(\mathbb{R}^+ \times M, U)$ and $T : M \to \mathbb{R}^+$, thus allowing for time–varying feedbacks u depending on the initial point of application), it is broad enough for our purposes.
Given a control quantization and an encoder, we have the diagram

$$\Sigma \xrightarrow{E} \mathcal{U} \xrightarrow{\Phi} \mathcal{D}(M),$$

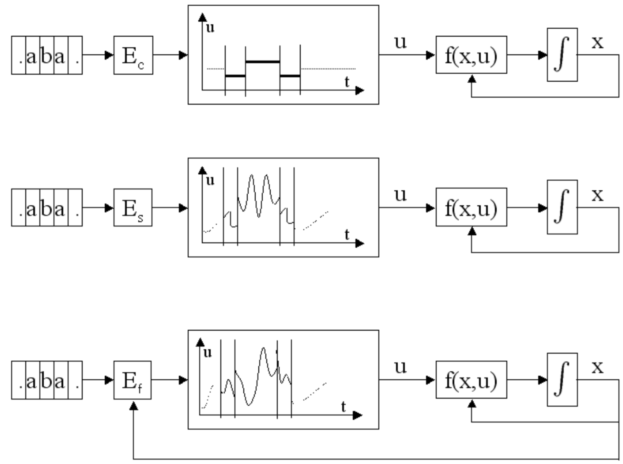

Fig. 2. Three examples of symbolic encoding of control: piecewise constant (top), piecewise smooth (middle), and feedback encoding.

which can be extended in an obvious way to

$$\Sigma^* \xrightarrow{E^*} \mathcal{U}^* \xrightarrow{\Phi^*} \mathcal{D}(M),$$

where Σ^* is the set of words formed with letters from the alphabet Σ, thus providing an action of the monoid Σ^* on M. Given a quantization $\mathcal{U}$ and an initial point x_0, let $R(\mathcal{U}, x_0)$ denote the reachable set from x_0 under $\mathcal{U}$, i.e. the Σ^*–orbit for the action defined via $\Phi^* \circ E^*$.
In general, being Σ^* just a monoid, the analysis of its action on $\mathcal{M}$ can be quite hard. However, in many cases, it is possible to define a group H as the quotient of Σ^* with respect to the kernel of the application $\Phi^* \circ E^*$ (or, if this is not possible for the whole Σ^*, by applying the quotient operation on particular submonoids). Once a group H and its action on the manifold M is obtained, the structure of the reachable set is completely determined when M can be identified with a Lie group G and it holds $\Phi^* \circ E^*(H) \subset G$. In this case, we have that if H is dense, discrete or finite then the reachable set will be correspondingly dense, discrete, or finite. Naturally, different choices of the symbolic encoding of control affect the structure of H. A particular appealing structure that one may want to achieve is a lattice, because representing and steering on lattices is particularly efficient. Another important concern is that the system maintains the possibility of approximating arbitrarily well all reachable equilibria in its state space. These requirements motivate the following definitions.
From now on we assume M endowed of a Riemannian metric indicated by d. Given a lattice Λ we define

$$\mathcal{S}(\Lambda) = \sup_{x \in M} \inf_{y \in \Lambda} d(x, y).$$

Definition 3: A control system (1) is *reticulable* if there exists a control quantization $\mathcal{U}$ such that, for every $x_0 \in$

M, $R(\mathcal{U}, x_0)$ is a lattice (finitely generated by commuting actions).

Notice that it may happen that reachable sets under symbolic encoding are discrete, but do not possess a lattice structure: this is e.g. the case for systems with drift as shown in the next section.

Definition 4: A control system (1) is:
– *ε–approachable* if there exists a control quantization $\mathcal{U}_\varepsilon$ such that, for every $x_0, y \in M$, there exists $x \in R(\mathcal{U}_\varepsilon, x_0)$ with $d(x, y) < \varepsilon$.
– *approachable* if it is ε–approachable for every $\varepsilon > 0$.
– *lattice-approachable* if for every $\varepsilon > 0$ there exists a control quantization $\mathcal{U}_\varepsilon$ for which the system is both reticulable and ε–approachable with commuting actions.

Remark: Notice that a system can be both reticulable and approachable without being lattice-approachable. Indeed the control quantization producing a lattice may fail to guarantee approachability and viceversa (this is the case for linear systems and the Dubins' car with constant control quanta, discussed later on).

It is clear that a lattice-approachable system can be conveniently quantized to approach any point in M from a given point x_0. Also, for a lattice-approachable systems it holds that, for every $\varepsilon > 0$, there exist $w_\varepsilon^1, \ldots, w_\varepsilon^{n_\varepsilon} \in \Sigma^*$ such that, for every $x_0 \in M$, the lattice $R(\mathcal{U}_\varepsilon, x_0)$ is generated by the action of $\Phi^* \circ E^*(w_\varepsilon^1), \ldots, \Phi^* \circ E^*(w_\varepsilon^{n_\varepsilon})$, where these actions commute. From this property we immediately get the following proposition, which explains our interest in lattice-approachable systems:

Proposition 1: If a control system (1) is lattice-approachable then for every $\varepsilon > 0$ and $x, y \in M$, there exist $m_1, \ldots, m_{n_\varepsilon}$ such that

$$d\left(\Phi^* \circ E^*\left((w_\varepsilon^1)_1^m \cdots (w_\varepsilon^{n_\varepsilon})^{m_{n_\varepsilon}}\right)(x), y\right) < \varepsilon.$$

The previous proposition implies that the planning problem is trivially solved in case of lattice-approachable systems.

III. LINEAR SYSTEMS

We first focus on quantizations with rational constant control laws proving various limitations. In particular we see that generic linear systems are not lattice-approachable by control quantization with rational constant control quanta. Then we pass to consider state–feedback control quanta showing their strength by proving that every reachable linear system is lattice–approachable from the origin.

Let us first introduce some more notations. Consider the reachable linear system: $\dot{x} = A\,x + B\,u$. If we use a constant control quantum $u(t, x) \equiv u, T \equiv \delta$ then we get the discrete time system:

$$x^+ = \tilde{A}\,x + \tilde{B}\,u, \tag{3}$$

where $\tilde{A} = e^{\delta A}$, $\tilde{B} = e^{\delta A} \int_0^\delta e^{-sA} B\, ds$. In case of $\tilde{A} = \tilde{B} = I$, the identity matrix, we have (cf. [13]):

Lemma 1: If $\mathcal{U} = \{v_1, \ldots, v_{n+1}\}$, where $v_1, \ldots, v_n$ are linearly independent, and w_i are the components of v_{n+1} w.r.t. to the other v_i's, then $R(\mathcal{U}, 0)$ is dense if and only if w_i is negative for all i and $1, w_1, \ldots w_n$ are linearly independent over $\mathbb{Z}$, that is $a_0 + a_1 w_1 + \cdots + a_n w_n = 0$, $a_i \in \mathbb{Z}$, if and only if $a_i = 0$ for all i.

For a scalar SISO system (3) reduces to: $x^+ = e^{ta} x + \frac{e^{at}-1}{a}\, b\, u$. To analyze the reachable set from zero the term $\frac{e^{at}-1}{a} b$ can be subsumed into u, hence we can neglect it. Defining $\lambda = e^{at}$ we get:

$$x^+ = \lambda x + u, \tag{4}$$

and from now on we treat directly this discrete time system. A key role is played by special class of algebraic integers called Pisot numbers. An algebraic integer $\lambda > 1$ is a Pisot number if its Galois conjugates, i.e. the other roots of the minimal polynomial, have norm strictly less than one. The most famous Pisot number is the golden number $g = \frac{1+\sqrt{5}}{2}$. Notice that the reachable set from 0 for a control quantization $\mathcal{U}$ is given by $R(\mathcal{U}, 0) = \{\sum_{i=0}^n u_i \lambda^i : n \in \mathbb{N}, u_i \in \mathcal{U}\}$, that are the evaluations at λ of polynomials with coefficients in $\mathcal{U}$. Pisot numbers are characterized by the fact that their annihilators, the set of polynomials with integer coefficients that vanish at the Pisot number, are recognizable by an automaton, see [17]. Roughly speaking, this implies that the reachable set $R(\mathcal{U}, 0)$ contains few points if $\mathcal{U}$ consists of integer constant control quanta.

Definition 5: A control quantization with control quanta (u, T), $T \equiv \delta$ and $u(t, x) \equiv u$, is called standard if the following holds: δ is equal for all control quanta, $0 \in \mathcal{U}$, $\exists 0 \neq u \in \mathcal{U}$ and finally if $u \in \mathcal{U}$ then $-u \in \mathcal{U}$.

For simplicity in the following we consider only standard control quantization.

Theorem 1: Consider the system (4) and standard control quantizations with rational constant control quanta. Then

1) For every $\lambda \geq 3$ there exists $\mathcal{U}$ such that $R(\mathcal{U}, 0)$ is a discrete set (but not a lattice).
2) For every $\lambda > 1$ (4) is approachable on any compact set. If $\lambda > 1$ is not Pisot then it is approachable from 0 on the whole $\mathbb{R}$.
3) For every $\lambda \geq 5$ compact K and $\varepsilon > 0$ there exists $\mathcal{U}$ with rational constant control quanta such that $R(\mathcal{U}, 0)$ is discrete and $\max_{x \in K} \min_{y \in R(\mathcal{U},0)} |x - y| < \varepsilon$.
4) For every compact K and $\varepsilon > 0$ there exists $\mathcal{U}$ with integer constant control quanta such that $R(\mathcal{U}, 0)$ is discrete and $\max_{x \in K} \min_{y \in R(\mathcal{U},0)} |x - y| < \varepsilon$ if and only if $\lambda > 1$ is a Pisot number.
5) (4) is not lattice-approachable.

In particular Theorem 1 implies that, a fortiori, a linear system in any dimension is not generically lattice approachable by rational constant control quanta. The proof of Theorem 1 relies on a deep analysis developed in [14], [18], [17]. We first recall some results from these papers restated in the following Proposition.

Proposition 2: Consider the system (4) with $\lambda > 1$ and let R_m be the reachable set from 0 with controls in $\{0, \pm 1, \cdots, \pm m\}$. Then

a) R_m is never dense for any $m \in \mathbb{N}$ if and only if λ is a Pisot number. Moreover, if λ is Pisot then the quantity $\sup_{y\in[-M,M]} \inf_{x\in R_m} |x-y|$ tends to zero as $m \to \infty$ for every M.

b) If λ is not a Pisot number then R_m is dense for every $m \geq 2(\lambda - \lambda^{-1})$.

c) If $m \leq (\lambda - 1)/2$ then R_m is not dense.

d) If $\lambda \geq 5$ then the maximum hole of R_1 is bounded by 1, that is $\max_{x\in R_1} \min_{x\neq y\in R_1} |x-y| \leq 1$.

Proof:[Theorem 1] From c) of Proposition 2 we get the conclusion 1. Indeed if $\lambda \geq 3$ then R_1 is discrete, thus the needed quantization consists of constant control quanta with $u = -1, 0, 1$.

Approachability from 0 is guaranteed by b) of Proposition 2 for non Pisot numbers and by a) for Pisot numbers. In both cases the needed quantization consists of constant control quanta with $u = -m, \dots, 0, \dots, m$, with m big enough. To conclude for every initial data in a compact set, consider the development in base λ of the initial point, details can be found in [14]. Assertion 4. follows similarly from a) of Proposition 2.

Consider conclusion 3. and fix a $\lambda \geq 5$. If λ is Pisot the conclusion is guaranteed by 4. Thus assume that λ is not Pisot. From c) of Proposition 2, R_1 is discrete. For every $q \in \mathbb{N}$ we consider the control set $U_q = \{0, \pm\frac{1}{q}\}$ and let R^q be the corresponding reachable set from 0. Since $R^q = \frac{1}{q} R_1$, R^q is a discrete set and, from d) of Proposition 2, we have that $\sup_{x\in[-M,M]} \inf_{y\in R^q} |x-y| \leq \frac{C(M)}{q}$ thus the quantity on the left-hand side is tending to zero (3. is proved).

Last assertion 5. follows directly from the fact that reachable sets are never lattices due to the presence of the drift term. ∎

Opposed to these results are the following showing the strength of state–feedback control quanta.

Theorem 2: A reachable linear system is always lattice-approachable from the origin by control quantization with rational state–feedback control quanta.

Proof: By standard change of variable and feedback, we can put the system in Jordan-Brunovsky normal form. That is made by blocks each of which is of the type:

$$\dot{x}_i = A_i x_i + u_i b_i,$$

where

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdot & \cdot & \cdots & 0 \end{pmatrix}, \qquad b_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

This system can be directly integrated and each constant control $\bar{v}_i$ applied for time t with initial point $(\bar{x}_{i,1} \dots, \bar{x}_{i,n_i})$ gives rise to the displacement vector $(\bar{x}_{i,2}t + \dots + \bar{x}_{i,n_i}\frac{t^{n_i-1}}{(n_i-1)!} + \bar{v}_i\frac{t^{n_i}}{n_i!}, \dots, \bar{x}_{i,n_i}t + \bar{v}_i\frac{t^2}{2}, \bar{v}_i t)$. Now to get reachable sets from the origin that are lattices it is enough to choose $\bar{v}_i$ and t rational. Moreover, the generated lattice can be arbitrarily fine. ∎

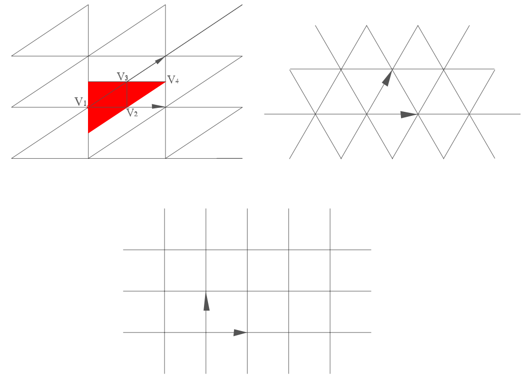

Fig. 3. Tessellation of the plane

IV. THE N–TRAILER AND CHAIN FORM SYSTEMS

We recall that the n–trailer system is formed by a car with n–trailers. The configuration of the car is identified by its baricenter position $(x, y) \in \mathbb{R}^2$, while the positions of trailers are given by angles θ_i between i–th trailer and $i+1$–trailer. For the case where the forward velocity of the vehicle is fixed, there are no trailers, and there are bounds on the trajectory curvature (this case is sometimes referred to as the Dubins' car), the equations are: $\dot{x} = \sin(\theta),\ \dot{y} = \cos(\theta),\ \dot{\theta} = u$, with $|u| \leq C$.

Proposition 3: The Dubins' car (car with no trailer) with constant control quantization is reticulable, approachable and ε–lattice-approachable only for $\epsilon > \min\{\frac{\pi}{4} + \frac{\sqrt{2}}{2C}, \frac{\pi}{3} + \frac{\sqrt{3}}{3C}\}$.

Proof: The action of a constant control quantum corresponds either to a rotation along a circle of radius $1/|u|$ if $|u| \neq 0$ or to translation along a straight line if $u = 0$. A quantization $\mathcal{U}$ thus leads to rotations with a finite number of possible radii and translations with a finite number of possible lengths. More precisely each constant control quantum acts as an element of $SE(2)$ on itself, after identifying $SE(2)$ with the semidirect product of $\mathbb{R}$ by S^1. Indicating by (r, θ) an element of $SE(2)$ ($r \in \mathbb{R}^2$, $\theta \in S^1$) the product rule is the following: $(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 + R(\theta_1) r_2, \theta_1 + \theta_2)$, where $R(\theta)$ is the rotation of angle θ. Each action of a constant control corresponds to a right multiplication by an element of $SE(2)$.

An equivalent problem is that motivated by rolling of a polyhedron over a plane, see [12]. If $R(\mathcal{U}, x_0)$ is a lattice then it is invariant for the subgroup of $SE(2)$ generated by a finite number of elements. It is well known that there are only two possible kinds of invariant lattices that correspond to a tessellation by squares or triangles, see Figure 3. Therefore the system is reticulable. On the other side, all generic choices of a finite number of constant control quanta generate a dense reachable set from any point, hence the system is approachable.

Since possible lattices generated by a quantization are those indicated above, th eproof is finalized. ■

The last part of the previous Proposition implies the following:

Theorem 3: The n–trailer system is not lattice-approachable by control quantization with constant control quanta.

On the other hand, in [13] we proved the following:

Theorem 4: A system in chained-form is lattice-approachable by control quantization with rational constant control quanta.

The proof relies on the following steps. 1) Find a natural base–fiber decomposition. This is immediate for chain–form systems and can be found by a general method looking for normal subgroups of $\mathcal{D}(\mathbb{R}^n)$ generated by $\Phi(\mathcal{U})$, see [19]. 2) Characterize the reachable set on the base part. 3) Find a group of generators for the subgroup of actions on the fiber. 4) Characterize the reachable set on the fiber part.

One of the key point is that we can write the group $\langle\Phi(\mathcal{U})\rangle$ as direct product of two subgroups whose action is additive on the base and, respectively, on the fiber. More generally we have the following:

Proposition 4: Consider a control system on $\mathbb{R}^n$. Assume that, for every control quantization $\mathcal{U}$, the group $\langle\Phi(\mathcal{U})\rangle$ can be written as the (semi)direct product of two subgroups each of which is additive on the corresponding either base or fiber. If the set of generators give rise to arbitrary small rational displacements, then the system is lattice–approachable.

It is well known that the n–trailer system can be put in chain form by a state–feedback, we thus get

Theorem 5: The n–trailer system is lattice-approachable by control quantization with state–feedback control quanta.

V. TRIANGULAR FORM SYSTEMS

A system is in *strictly triangular form* if it is written as

$$\begin{array}{rcl} \dot{x}_1 &=& \sum_{i=1}^p g_1^i(x_2,\dots,x_p)u_i \\ \dot{x}_2 &=& \sum_{i=1}^p g_2^i(x_3,\dots,x_p)u_i \\ &\vdots& \\ \dot{x}_{p-1} &=& \sum_{i=1}^p g_{p-1}^i(x_p)u_i \\ \dot{x}_p &=& \sum_{i=1}^p g_p^i u_i \end{array} \tag{5}$$

with $x = [x_1, x_2, \dots, x_p] \in \mathbb{R}^{n_1+n_2+\dots+n_p} = \mathbb{R}^n$, $u = [u_1, ..., u_p] \in \mathbb{R}^{n_p}$, $n_p = p$.

We recall that nilpotent systems are feedback equivalent to strictly triangular systems with polynomial coefficients (g_j^i) (see [20]) while solvable systems are state equivalent to strictly triangular form systems with no restriction on the coefficients g_j^i (see [21]). A crucial property of strictly triangular form systems is that they can be integrated by quadratures (that is, by simple monovariate integration formulae).

The approach of quantization for strictly triangular form systems was proposed in [15] where a control quantization by "loop" control quanta was shown to efficiently solve the steering problem. By loop control quantum we mean a control quantum for which the x_p variables of the systems undergo a closed path in $\mathbb{R}^{n_p}$.

A particular choice of loop control quantum is given by a piecewise–constant control quantum of the type $u(t) = [u_1, u_2] = u_1 \circ u_2 \circ \bar{u}_1 \circ \bar{u}_2$ where $u_i : [0,\tau] \to \mathbb{R}^{n_p}$ are constant, $\bar{u}_i = -u_i$ for $i = 1, 2$ and $\circ$ denotes the concatenation operation. Hence we let $\mathcal{U} = \mathcal{U}_{p+1}$ be a control quantization by constant control quanta and $\mathcal{U}_p^\star = [\mathcal{U}_{p+1}^\star, \mathcal{U}^s tar_{p+1}]$ the group generated by the loop control quanta. Notice that if $u, u_1, u_2 \in \mathcal{U}_{p+1}$ then $u[u_1, u_2]\bar{u} \in \mathcal{U}_p^\star$ indeed

$$u[u_1,u_2]\bar{u} = u \circ (u_1 \circ u_2 \circ \bar{u}_1 \circ \bar{u}_2)\bar{u} = u_1^u \circ u_2^u \circ \bar{u}_1^u \circ \bar{u}_2^u$$

where $u_1^u = u \circ u_1 \circ \bar{u}$ and $\bar{u}_1^u = u \circ \bar{u}_1 \circ \bar{u}$. Thus $\mathcal{U}_p^\star$ is a normal subgroup. Analogously we let $\mathcal{U}_{p-1}^\star = [\mathcal{U}_p^\star, \mathcal{U}_p^\star] \subset \mathcal{U}_p^\star$ be a control quantization for which x_p and x_{p-1} undergo a closed path in $\mathbb{R}^{n_{p-1}+n_p}$. We proceed until a flag of control quantizations $\mathcal{U}_p^\star \supset \mathcal{U}_{p-1}^\star \supset \cdots \supset \mathcal{U}_2^\star$ is given such that $\mathcal{U}_i^\star = [\mathcal{U}_{i+1}^\star, \mathcal{U}_{i+1}^\star]$ makes the variables $x_p, \dots, x_i$ undergo a closed loop. The following property holds for such a choice of control quantization.

Proposition 5: For each $i = 2, \dots, p+1$ we have that $\mathcal{U}_i^\star$ provides an additive action on $\mathbb{R}^{n_{i-1}}$.

Thanks to additivity property we have:

Definition 6: To each $u \in \mathcal{U}_i^\star$ there correspond a vector $\delta_i \in \mathbb{R}^{n_{i-1}}$, called quantum displacement, and a net motion on the x_{i-1} variables, given by $x_{i-1}^+ = x_{i-1} + \delta_i$. We denote by N_i the cardinality of $\mathcal{U}_i^\star$, by δ_i^j, $j = 1, \dots, N_i$, the quantum displacements corresponding to the control quantum $u_i^j \in \mathcal{U}_i^\star$ and by $Q_i = \{\delta_i^j,\ u_i^j \in \mathcal{U}_i^\star\}$ the whole set of quantum displacements corresponding to controls in $\mathcal{U}_i^\star$.

Observe that it may happen that the elements of Q_i are not linearly independent. However, if $N_i \geq n_{i-1}$, in generic hypothesis, we can assume that $\mathsf{span}(Q) = \mathbb{R}^{n_{i-1}}$. Moreover, by construction, we have that $N_{i+1} = N_i(N_i - 1)/2$.

Proposition 6: A controllable system in triangular form with control quantization is approachable by constant control quanta.

Proof: To prove the proposition it is sufficient to guarantee that, for each i, there are sufficiently many elements in Q_{i+1} to span $\mathbb{R}^{n_i}$ and that the subgroup generated by Q_{i+1} is a dense subset of $\mathbb{R}^{n_i}$. Therefore, by Lemma 1, it is sufficient to choose $\mathcal{U}$ such that the following are satisfied: **1)** $N_i > n_{i-1} + 1$ for all $i = 2, \dots, p+1$ **2)** up to reordering the indices, $\delta_i^{n_{i-1}+1} = \sum_{j=1}^{n_{i-1}} a_j \delta_i^j,\ a_j < 0,\ a_j \notin \mathbb{Q}$, for $i = 2, \dots, p+1$. Here condition **1)** guarantees that the control set is sufficiently reach and condition **2)** guarantees that the reachable set is dense. ■

Proposition 7: A strictly triangular system, where the g^i's are polynomials with rational coefficients, is lattice-approachable with rational constant control quanta.

Proof: Consider a set $\mathcal{U}_{p+1}$ of rational constant control quanta. Then Q_{p+1} is given by $\{\sum_{i=1}^p g_p^i u_i, u = [u_1, \dots, u_p] \in \mathcal{U}_{p+1}\}$. Assume that $g_p^i \in \mathbb{Q}$ then the group generated by Q_{p+1} is a lattice Λ_p with size $\mathcal{S}(\Lambda_p)$ that can be made as small as desired by tuning on the controls $\mathcal{U}_{p+1}$. (Notice that with integer constant control quanta the system would be only reticulable with size constrained by the g_p^i.) Consider now $\mathcal{U}_p^\star$ and the corresponding set $Q_p = \{\int \sum_{i=1}^{p} g_{p-1}^i(x_p)u_i(t),\ \ u(t) \in \mathcal{U}_p^\star\}$. The $g_{p-1}^i(x_p)$ are polynomials with rational coefficients in x_p. Then the quantum displacements in Q_p are vectors with rational coefficients and the set of reachable configurations for x_{p-1} are a lattice with arbitrary size (depending on the rational constant control quanta u_i). The results for x_{p-1} extend in analogous way to x_i for all $i = 1, \ldots, p-2$. ■

VI. CONCLUSIONS

In this paper we have considered control encoders that allow to build a control language by which steering of complex dynamical systems is efficiently solvable. The key property of the system this approach relies upon is lattice-approachability. We have shown that such property is achieved by different important classes of systems, depending upon the different encoding schemes that are allowed. Among the most expressive encoding schemes considered in this paper are feedback encoders, by which a very large class of systems is lattice-approachable. Investigations are under way as to which conditions can guarantee lattice-approachability of a dynamic system under different encoding schemes.

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