OBSERVABILITY AND NONLINEAR OBSERVERS FOR MOBILE ROBOT LOCALIZATION [1]

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Abstract: In this paper, the problem of localizing a nonholonomic vehicle moving in an unstructured environment is addressed by the design of a novel nonlinear observer, based on suitably processed optical information. The localization problem has been shown to be intrinsically nonlinear, indeed the linear approximation of the system has different structural properties than the original model. The proposed nonlinear observer allows the localization of nonholonomic mobile robots by using an estimated expression of the extended output Jacobian, local practical stabilization of the observation error dynamics is also ensured. A singularity-avoidance exploration strategy is also included to deal with the output Jacobian singularities crossing.

Keywords: Nonholonomic robot dynamics, Nonlinear observability, Nonlinear observers, Local practical stability, Singularities avoidance, Simulation results

1. INTRODUCTION

The problem of mobile robot localization using optical information is intrinsically nonlinear (Bicchi *et al.*, 1998), in fact linearized approximations are non–observable, while tools from differential–geometric nonlinear system theory prove the possibility of reconstructing the position and orientation of the vehicle and the position of the obstacles in the environment. The localization problem can be embedded in the more general problem of current state estimation of a dynamical system using input-output measurements. For smooth nonlinear systems, a classical approach to design an observer is to apply the extended Kalman filter, see (J. Borenstein and Feng, 1996; Gelb, 1974; Misawa and Hedrick, 1989).

Another way of designing an observer is to transform the original nonlinear system into another one for which the design is known. Transformations, which have been proposed in the literature, are the system immersion (Fliess and Kupka, 1983) which permits to obtain a bilinear system if the observation space is finite dimensional, and the linearization by means output injection (Krener and Isidori, 1983; Krener and Respondek, 1985; Levine and Marino, 1986) assuming that particular differential–geometric conditions on the system vector fields are verified. Rank conditions under which the dynamics of the observation error is linear, i.e. the original system can be transformed into the observer canonical form, are also investigated in (Xia and Gao, 1989). Results on bilinear observers are presented in (F. Celle and Kazakos, 1986; Hammouri and Gauthier, 1988). Extension of the Luenberger filter in a nonlinear setting, by using the time derivatives of the input, has been proposed in (Zeitz, 1987). A nonlinear observer and its practical implementation have been presented in Gauthier et al. (J.P. Gauthier and Othman, 1992; Bonard and Hammouri, 1991), in which the first step is writing the input affine nonlinear system in a

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so-called normal observation form. However, this form requires that the trivial input is an universal input (Bonard *et al.*, 1995) for the system, and also that a diffeomorphism can be constructed using the Lie derivatives of the output along the drift nonlinear term. In our problem, both these conditions are violated by the mobile robot kinematics.

In this paper, a local nonlinear observer for mobile robot localization is designed, inspired by the work of Teel and Praly (Teel and Praly, 1994). The basic idea is to use output derivatives, which are elements of the observability space associated to the nonlinear measurement process, in the assumption that a rank condition on the observability matrix is satisfied for the considered input function. The observer design is based on the concept of Extended Output Jacobian (EOJ) matrix, which is the collection of the covectors associated to the considered elements of the observability space, i.e. the output and its derivative. Then, the output derivatives are estimated by using high-pass filters. Local practical stability of the observation error dynamics is guaranteed since the introduced persistent perturbations can be made arbitrarily small. A singularity-avoidance exploration task is also addressed to deal with the singularity occurrences in the EOJ matrix.

The paper is organized as follows. In sect. 2 the localization problem is re-casted as one in nonlinear observability. In sect. 3, the novel nonlinear observer for nonholonomic vehicle localization is designed. In sect. 4, conditions on the input vector which ensure singularity avoidance are also derived. Sect. 5 reports simulation results in both the cases of output measurements affected by bounded noise, and of singularity-avoidance exploration. Finally, in sect. 6 the major contribution of the paper is summarized.

Notation - $\|\cdot\|$ denotes the Euclidean norm for a vector and an induced natural norm for a matrix. $(\Re^n)^*$ is the dual space of the Euclidean space $\Re^n$. Given a scalar function $f : \Re^n \to \Re$, df denotes the associated exact differential. Given a vector field $\mathbf{f} : \Re^n \to \Re^p$, $d\mathbf{f}$ denotes the associated Jacobian matrix.

2. OBSERVABILITY

Consider a system comprised of a mobile vehicle, such as a robotic rover in a planetary exploration mission, which moves in an unknown environment with the aim of localizing itself and the environment features (in the rover case, e.g., rocks and geological formations). Among the features that the sensor head detects in the robot environment, we will distinguish between those belonging to objects with unknown positions (which we shall call *targets*), and those belonging to objects whose absolute position is known, which will be referred to as *markers*. The vehicle is a kinematic unicycle whose dynamics are slow enough to be neglected:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \\ \dot{\theta} \\ \dot{\xi}_1 \\ \dot{\zeta}_1 \\ \vdots \\ \dot{\xi}_N \\ \dot{\zeta}_N \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u_2, \qquad (1)$$

where ξ, ζ represent the coordinates of the position of the vehicle with respect to some arbitrary fixed reference frame, and θ is the angle between the ξ axis and the direction of motion of the rover; ξ_i, ζ_i are the position coordinates (in the same reference) of the i–th target; u_1 is the vehicle forward velocity, while u_2 is its angular velocity. Notice that the system is in the standard form of nonlinear systems which are linear in control, i.e. $\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2$. The measurement process is modeled by N equations for target observations in the form

$$y_i = h_i(\mathbf{x}) = \pi + \text{atan2}(\zeta - \zeta_i, \xi - \xi_i) - \theta, \quad (2)$$
$$i = 1, \ldots, N,$$

and by M further measurements relating to markers, whose absolute position $(\bar{\xi}_i, \bar{\zeta}_i)$ is known, as

$$y_i = h_i(\mathbf{x}) = \pi + \text{atan2}(\zeta - \bar{\zeta}_i, \xi - \bar{\xi}_i) - \theta, \quad (3)$$
$$i = N+1, \ldots, N+M.$$

The localization problem is therefore stated, in terms of the above model, as a problem of reconstructing the initial state $\mathbf{x}(0)$ of the system Eq.(1) from measurement of the $M + N$ output angles $\mathbf{y}(t)$.

¿From the linear approximation of the system (about an arbitrary initial state $\hat{\mathbf{x}}$ and zero control), it can be easily seen (Bicchi *et al.*, 1998) that the linearized system is always unobservable if there are targets ($N \neq 0$), and can only be observable if there are at least 3 markers ($M \geq 3$). This result contrasts with the common practice in navigation and surveying of making the point by triangulation. It has been proven in (Bicchi *et al.*, 1998), by using the codistribution associated to the observability space, that if, at least, two markers are available, nonlinear observability of the location of the vehicle and of targets follows.

3. NONLINEAR OBSERVERS FOR LOCALIZATION

In this section, a novel nonlinear observer for mobile robot localization is presented, by considering the robot model (1) with $N = 0$ (no targets) and two markers $M = 2$, i.e. $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{u})$, where $\mathbf{x} = [\xi\ \zeta\ \theta]^T \in \Re^2 \times S^1$ is the state vector, and

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} \cos\theta u_1 \\ \sin\theta u_1 \\ u_2 \end{pmatrix} \quad , \tag{4}$$

with output measurements: $\mathbf{y} = \mathbf{h}(\mathbf{x}) = [y_1\ y_2]^T$, $y_i = h_i(\mathbf{x}) = \pi + \text{atan2}(\zeta - \bar{\zeta}_i, \xi - \bar{\xi}_i) - \theta,\ i = 1, 2$, as in Eq. (3).

The following observer is introduced for the nonholonomic vehicle:

$$\dot{\hat{\mathbf{x}}} = \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{P}\,(\mu - \hat{\mu}) \quad , \tag{5}$$

where $\mu = [\mathbf{y}^T\ \dot{\mathbf{y}}^T]^T \in \Re^4$, and $\hat{\mu}(\hat{\mathbf{x}}) = [\mathbf{h}(\hat{\mathbf{x}})^T\ \dot{\mathbf{h}}(\hat{\mathbf{x}})^T]^T$, the design parameter $\mathbf{P} \in \Re^{3\times 4}$ is chosen as $\mathbf{P} = (\mathbf{Q} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}}))\,\mathbf{J}^+$, being $\mathbf{Q}$ a positive definite matrix, the Extended Output Jacobian (EOJ) matrix $\mathbf{J}$ is defined as:

$$\mathbf{J}(\hat{\mathbf{x}}, \mathbf{u}) = d\,\hat{\mu}(\hat{\mathbf{x}}) = \begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \\ \frac{\partial \dot{\mathbf{h}}}{\partial \mathbf{x}} \end{bmatrix} \in \Re^{4\times 3} \quad . \tag{6}$$

If $\mathbf{J}$ is full-rank, $\mathbf{J}^+ = (\mathbf{J}^T\,\mathbf{J})^{-1}\,\mathbf{J}^T$ is its left pseudo-inverse. We now give a preliminary motivation of the introduced observer, based on the Implicit Function Theorem. Rearrange the output equation and its derivative in the form:

$$\mathbf{F}(\mathbf{z}, \mathbf{x}) = \begin{pmatrix} \mathbf{y} - \mathbf{h}(\mathbf{x}) \\ \dot{\mathbf{y}} - \dot{\mathbf{h}}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{y} - h(\mathbf{x}) \\ \dot{\mathbf{y}} - \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}, \mathbf{u}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where we have introduced the variable $\mathbf{z} = [\mu^T\ \mathbf{u}^T]^T$. Since $\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = -\mathbf{J}$, if the matrix $\mathbf{J}$ is full-rank in a fixed point $(\mathbf{z}^0, \mathbf{x}^0)$, then, by using the Implicit Function Theorem, there exist two neighborhoods A^0 of $\mathbf{z}^0$, B^0 of $\mathbf{x}^0$, and an unique map $\mathbf{l} : A^0 \to B^0$, with $\mathbf{l}(\mathbf{z}) = \mathbf{x}$, such that $\mathbf{F}(\mathbf{z}, \mathbf{l}(\mathbf{z})) = 0$. Hence, the full-rank of matrix $\mathbf{J}$ implies the local existence of a static map which relates the state with the output, its time derivative, and the input. Since the analytical expression of the map $\mathbf{l}(\mathbf{z})$ can not, in general, be derived, in the nonlinear observer (5) we have chosen to consider directly the implicit condition $\mathbf{F}(\mathbf{z}, \mathbf{x}) = 0$ and linearize this expression with respect to the actual estimate $\hat{\mathbf{x}}$, as will be clear soon. The following proposition formally summarizes the properties of the proposed observer in the case of exact measurements of the output derivatives.

Proposition 1. Assume exact measurements of the output derivative, and that the EOJ matrix is full rank, i.e. $\text{rank}(\mathbf{J}) = 3$, $\forall \mathbf{x},\ \mathbf{u}$ of interest. Then the equilibrium $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} = 0$ of the observation error dynamics constructed by using the observer in Eq. (5) is locally asymptotically stable.

Proof:
The observation error has the following dynamics:

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{g}(\mathbf{x}, \mathbf{u}) + \mathbf{P}\,(\mu - \hat{\mu}) \\ &= \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}})\right)\mathbf{e} - \mathbf{P}\mathbf{J}(\hat{\mathbf{x}}, \mathbf{u})\,\mathbf{e} + \mathbf{P}O_\mu^2(\|\mathbf{e}\|) \\ &- O_\mathbf{g}^2(\|\mathbf{e}\|) \end{aligned} \tag{7}$$

where we have linearized the vector field $\mathbf{g}(\mathbf{x}, \mathbf{u})$ and the extended output vector μ around the actual state estimate, i.e. $\mathbf{g}(\mathbf{x}, \mathbf{u}) - \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}}) + O_\mathbf{g}^2(\|\mathbf{g}\|)$ and $\mu - \hat{\mu} = \mathbf{J}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{x} - \hat{\mathbf{x}}) + O_\mu^2(\|\mathbf{e}\|)$.

Consider the quadratic Lyapunov function candidate $V = \frac{1}{2}\mathbf{e}^T\,\mathbf{e}$, its time derivative results:

$$\begin{aligned} \dot{V} &= \mathbf{e}^T\,\dot{\mathbf{e}} \\ &= \mathbf{e}^T\,[(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}}))\mathbf{e} - \mathbf{P}\mathbf{J}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{e} + \mathbf{P}O_\mu^2(\|\mathbf{e}\|) \\ &- O_\mathbf{g}^2(\|\mathbf{e}\|)] = -\mathbf{e}^T\,\mathbf{Q}\mathbf{e} + \mathbf{P}O_\mu^3(\|\mathbf{e}\|) - O_\mathbf{g}^3(\|\mathbf{e}\|) \quad . \end{aligned} \tag{8}$$

Hence, the function $\dot{V}$ is negative definite in a neighborhood of the origin $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} = 0$. □

Remark 1. To design the matrix $\mathbf{P}$, it has been used the pseudo-inverse of $\mathbf{J}$, the study of the locus of singularities is then necessary to investigate system trajectories and particular inputs which can cause loss of rank in $\mathbf{J}$. We first remark that in the case of an uniformly observable SISO (Single Input Single Output) nonlinear system (Teel and Praly, 1994; Isidori, 1995), given the map $[y\ \dot{y}\ \ldots y^{(n-1)}]^T = \Phi(\mathbf{x}, \mathbf{v})$, being $\mathbf{x} \in \Re^n$ the state vector, and $\mathbf{v} = [u\ \dot{u}\ \ldots u^{(n-2)}]^T \in \Re^{n-1}$ the extended input vector, the matrix $\mathbf{J}$, which is is related to the expression of the tangent map of $\Phi(\mathbf{x}, \mathbf{v})$, is nonsingular, for definition, for each $(\mathbf{x}, \mathbf{v}) \in \Re^n \times \Re^{n-1}$. Thus, uniformly observability is a sufficient condition to design the observer (5).

Remark 2. Assume that the input function $\mathbf{u}(t),\ t \geq t_0 \geq 0$ of the vector field in Eq. (4) is real analytic. The matrix $\mathbf{J}$ is full-rank in a open neighborhood $\mathbf{X}^0 \times \Re^2$, if the system is locally observable at every $\mathbf{x}^0 \in \mathbf{X}^0$ in the interval $[t_0,\ T]$, for some $T > t_0$, i.e. the observability matrix for the particular input $\mathbf{u}(t),\ t \in [t_0,\ T]$ is nonsingular in a neighborhood of $\mathbf{x}^0$ (Xia and Gao, 1989; Zeitz, 1987).

Example 1. In the case of a holonomic vehicle described by the equation: $\dot{\mathbf{x}} = \mathbf{u},\ \mathbf{x}, \mathbf{u} \in \Re^2$, assume, for example, that it is possible to measure only the distance: $y = \frac{1}{2}(x_1^2 + x_2^2)$. The EOJ matrix results:

$$\mathbf{J}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} x_1 & x_2 \\ u_1 & u_2 \end{bmatrix} \quad , \qquad (9)$$

the locus of singularities is given by the set $S = \{(\mathbf{x}, \mathbf{u}) \in \Re^2 \times \Re^2 : \; x_1 u_2 - x_2 u_1 = 0\}$. The above analysis indicates that the holonomic vehicle is not uniformly observable, and suggests to choose the input $\mathbf{u}$ such that, at each time instant, the point $(\mathbf{x}, \mathbf{u})$ does not lie on the manifold S.

To prove the practical stability of the observer in Eq. (5) in case of estimated output derivatives, we first remind the following result on the total stability of nonlinear systems, see for example (Isidori, 1995):

Proposition 2. Consider the nonlinear system $\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t)$, where $\mathbf{x} \in \Re^n$, $\mathbf{u}(t) \in \Re^m$, and $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$ are smooth vector fields. Assume that the origin of $\mathbf{x} = \mathbf{f}(\mathbf{x})$ is a locally asymptotically stable equilibrium. Then $\forall \epsilon > 0$, there exist $\delta_1 > 0$ and $K > 0$ such that if $\|\mathbf{x}(0)\| < \delta_1$ and $\|\mathbf{u}(t)\| < K, \; t \geq t^0 \geq 0$, the solution $\mathbf{x}(t, t^0, \mathbf{u})$ of $\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t)$ satisfies the condition: $\|\mathbf{x}(t, t^0, \mathbf{u})\| < \epsilon, \; t \geq t^0 \geq 0$.

Now we state the following.

Proposition 3. Consider the nonlinear observer:

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{P} \begin{bmatrix} \mathbf{h}(\hat{\mathbf{x}}) \\ \dot{\mathbf{h}}(\hat{\mathbf{x}}) \end{bmatrix} + \mathbf{P} \begin{bmatrix} \mathbf{y} \\ \psi \end{bmatrix} \\ \dot{\psi} &= \frac{1}{T}(-\psi + \dot{\mathbf{y}}) \quad , \end{aligned} \qquad (10)$$

where T is a sufficiently small positive constant. Assume that the EOJ matrix is full rank, i.e. $\text{rank}(\mathbf{J}) = 3, \; \forall \mathbf{x}, \; \mathbf{u}$ of interest. Then the equilibrium $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} = 0$ of the observation error dynamics constructed by using the observer in Eq. (10) is locally practically stable, i.e. $\forall \epsilon_{\mathbf{e}} > 0$ there exist $\delta_1 > 0$, $K > 0$, and T which depends on K, such that if the initial observation error is sufficiently small, i.e $\|\mathbf{e}(0)\| < \delta_1$, then $\|\epsilon^*(t)\| < K, \; t \geq 0$, being $\epsilon^* = [0^T \; \epsilon^T]^T$, $\epsilon = \dot{\mathbf{y}} - \psi$, and the observation error satisfies the condition $\|\mathbf{e}(t, 0, \epsilon^*)\| < \epsilon_{\mathbf{e}}, \; t \geq 0$.

Proof:
The dynamics of the observation error $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$ results: $\dot{\mathbf{e}} = -\mathbf{Q}\mathbf{e} - \mathbf{P}\epsilon^* + \mathbf{P}O_\mu^2(\|\mathbf{e}\|) - O_{\mathbf{g}}^2(\|\mathbf{e}\|)$, where $\epsilon^* = [0 \; \epsilon]^T$, $\epsilon = \dot{\mathbf{y}} - \psi$ is the introduced error due to the output derivative estimation. The output derivatives error ϵ can be reduced to an arbitrarily small perturbation if the constant T is sufficiently small. In fact, assumed $\psi(0) = 0$, the Laplace transform of ϵ results: $\epsilon(s) = \dot{\mathbf{Y}}(s) - \frac{s}{1+T\,s}\mathbf{Y}(s) + \frac{\mathbf{y}(0^+)}{1+T\,s} = T\frac{s}{1+T\,s}\dot{\mathbf{Y}}(s)$, where $\mathbf{Y}(s)$ denotes the output Laplace transform, and (with an abuse of notation) $\dot{\mathbf{Y}}(s) = s\,\mathbf{Y}(s) - \mathbf{y}(0^+)$ denotes the output derivative Laplace transform. $\forall t \geq 0$, $\epsilon(t) = T\,\dot{\mathbf{y}}_f(t)$, where $\dot{\mathbf{y}}_f(t)$ is the output derivative filtered by $\frac{s}{1+Ts}$. Since the high-pass filter $\frac{s}{1+Ts}$ is Bounded Input Bounded Output

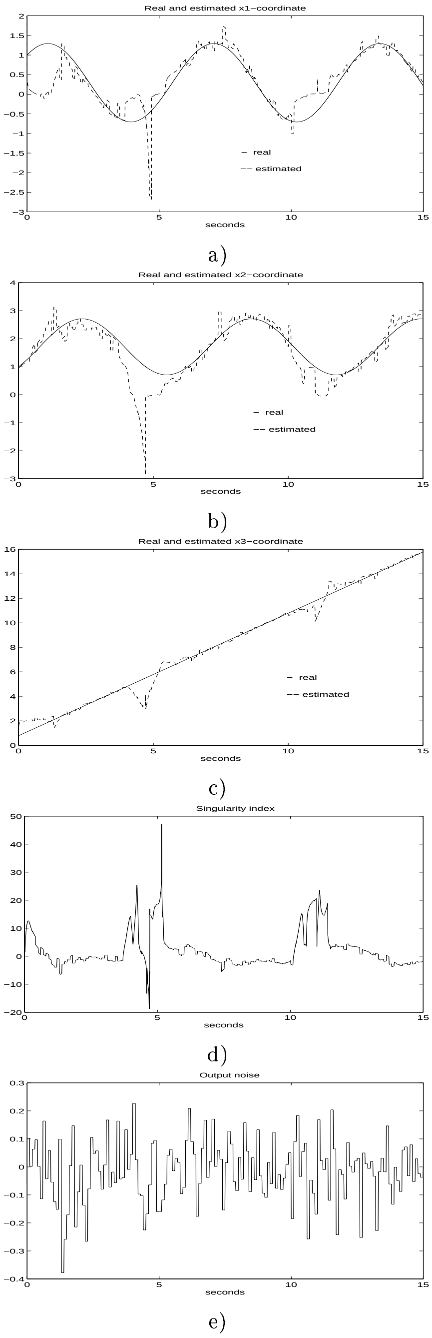

Fig. 1. Nonholonomic vehicle (constant input): state estimation in presence of output measurements noise. a) real and estimated x_1 component, b) real and estimated x_2 component, and c) real and estimated x_3 component of the state vector. d) Singularity index of the EOJ matrix: $\log w(\hat{\mathbf{x}}, u)$; e) output noise.

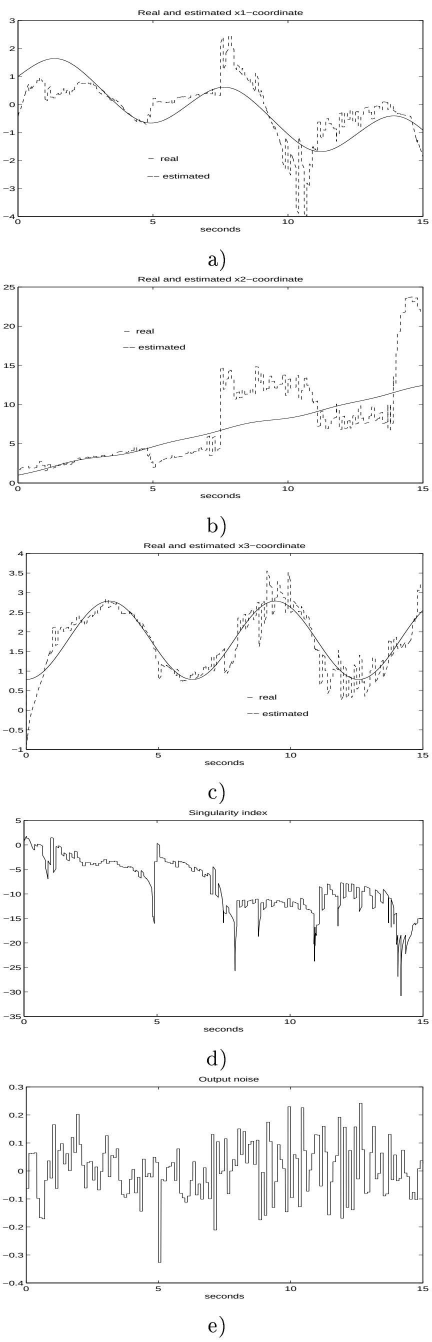

Fig. 2. Nonholonomic vehicle (time-varying input): state estimation in presence of output noise. a) real and estimated x_1 component, b) real and estimated x_2 component, and c) real and estimated x_3 component of the state vector. d) Singularity index of the EOJ matrix: $\log w(\hat{\mathbf{x}}, u)$; e) output noise.

(BIBO), there exists $M_{\dot{\mathbf{y}}} > 0$ such that $\|\dot{y}_f(t)\| < M_{\dot{\mathbf{y}}}$, $\forall t \geq 0$. Hence $\forall \bar{\epsilon} > 0$, by choosing $T < \frac{\bar{\epsilon}}{M_{\dot{\mathbf{y}}}}$, then $|\epsilon(t)| < T\, M_{\dot{\mathbf{y}}} < \bar{\epsilon}$, $\forall t \geq 0$. Let us consider the nonlinear system $\dot{\mathbf{e}} = -\mathbf{Q}\mathbf{e} - \mathbf{P}\epsilon^* + \mathbf{P}O^2_{\mu}(\|\mathbf{e}\|) - O^2_{\mathbf{g}}(\|\mathbf{e}\|)$. If $\epsilon^*(t) = 0,\ t \geq 0$ the origin $\mathbf{e} = 0$ (of the unperturbed system) is locally asymptotically stable. Prop. 2 indicates that $\forall \epsilon_{\mathbf{e}} > 0$ there exist $\delta_1 > 0$ and $K > 0$ such that if the initial observation error is sufficiently small, i.e $\|\mathbf{e}(0)\| < \delta_1$ and choosing $\bar{\epsilon} = K$, since there exists T such that $\|\epsilon^*\| = \|\epsilon\| < \bar{\epsilon} = K$, then the observation error satisfies the condition $\|\mathbf{e}(t, 0, \epsilon^*)\| < \epsilon_{\mathbf{e}},\ t \geq 0$.

□

Finally, some robustness considerations are briefly discussed in order to address the realistic case of measurements affected by noise. Assume that the output measurements $\mathbf{y}_m$ are given by: $\mathbf{y}_m = \mathbf{y} + \delta\mathbf{y}$, where $\delta\mathbf{y}$ is a bounded additive noise, and $\mathbf{y}$ is the real output. In the case of exact measurement of the output derivatives, since in the observer is used the extended output vector μ, the measured vector is indeed $\mu_m = \mu + \delta\mu$, where the perturbation results $\delta\mu = [\delta\mathbf{y}^T\ \delta\dot{\mathbf{y}}^T]^T$, being $\delta\dot{\mathbf{y}}^T$ the measurement error in the output derivative. The designed observer is:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{P}\,(\mu_m - \hat{\mu}) \\ &= \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{P}\,(\mu + \delta\mu - \hat{\mu}) \quad . \end{aligned} \tag{11}$$

Consider the quadratic Lyapunov function candidate $V = \frac{1}{2}\mathbf{e}^T\mathbf{e}$, by repeating the same calculations of Prop. 1's proof, its time derivative results: $\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} + \mathbf{e}^T\mathbf{P}\delta\mu + \mathbf{P}O^3_{\mu}(\|\mathbf{e}\|) - O^3_{\mathbf{g}}(\|\mathbf{e}\|)$. In a neighborhood of the origin $\mathbf{e} = 0$ contained in the region of attraction of the unperturbed observer (i.e. without output noise), the time derivative $\dot{V}$ is negative, at least, in the region: $\|\mathbf{e}\| > \frac{\|\mathbf{P}\|\,\|\delta\mu\|}{\lambda_m}$, where λ_m is the minimum eigenvalue of $\mathbf{Q}$. Hence the observation error is bounded in case of small bounded perturbations $\delta\mu$. The same considerations apply in the case of output derivatives estimated via high-pass filters. It is easy to verify that in a neighborhood included in the region of attraction, the time derivative $\dot{V}$ is negative if $\|\mathbf{e}\| > \frac{\|\mathbf{P}\|\,(\|\delta\mu\| + \|\epsilon^*\|)}{\lambda_m}$, where ϵ^* is the arbitrary small persistent perturbation introduced by the high-pass filters, and $\delta\mu = [\delta\mathbf{y}^T\ \delta\psi^T]^T$ is the perturbation due to output noise, being $\delta\psi(s) = \frac{s}{1+T\,s}\,\delta\mathbf{y}(s) + \frac{\delta\mathbf{y}(0^+)}{1+T\,s}$ the added measurement noise. The observation error is then bounded in case of small bounded perturbations.

4. SINGULARITY-AVOIDANCE STRATEGY

Consider the singularity index associated to the EOJ matrix defined as $\log w(\hat{\mathbf{x}}, \mathbf{u})$, where the introduced singularity function is $w(\hat{\mathbf{x}}, \mathbf{u}) = \sqrt{\det\,(J^T\,J)}$. The quality of the current state estimation depends on the actual value of $w(\hat{\mathbf{x}}, \mathbf{u})$. A possible singularity-avoidance strategy to guarantee a well-conditioned EOJ matrix is to impose the following input derivative expression

$$\dot{\mathbf{u}} = k^*\,(\frac{\partial w(\hat{\mathbf{x}}, \mathbf{u})}{\partial \mathbf{u}})^T,\ k^* > 0 \quad . \tag{12}$$

where $\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}} \in (\Re^2)^*$. If this is the case, the time derivative of the singularity function $w(\hat{\mathbf{x}},\mathbf{u})$ results:

$$\dot{w}(\hat{\mathbf{x}},\mathbf{u}) = \left(\frac{\partial w}{\partial \mathbf{x}}(\hat{\mathbf{x}},\mathbf{u})\right)\dot{\hat{\mathbf{x}}} + k^* \left\|(\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}})^T\right\|^2 \quad ,(13)$$

which is certainly positive if k^* is chosen sufficiently large. Notice that another way to obtain an increasing singularity function is to choose

$$\dot{\mathbf{u}} = k^* \, sgn\left((\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}})^T\right), \; k^* > 0 \quad , (14)$$

where, in general, given $\mathbf{x} \in \Re^n$, $sgn(\mathbf{x}) = [sgn(x_1), \ldots, sgn(x_n)]^T$, being $sgn(x_i)$ the sign function, which values 1 if $x_i > 0$, -1 if $x_i < 0$, and $sgn(0) = 0$. This choice yields $\dot{w}(\hat{\mathbf{x}},\mathbf{u}) = \left(\frac{\partial w}{\partial \mathbf{x}}(\hat{\mathbf{x}},\mathbf{u})\right)\dot{\hat{\mathbf{x}}} + k^* \sum_i^2 \left|(\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}})_i\right|$ where $(\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}})_i$ denotes the $i-th$ element of the covector $\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}}$. Fixed an arbitrary input initial condition, a *first technique* to satisfy the above equation is to compute the input function through integration from Eqs. (12) or (14). In particular in the latter case, since the input derivative $\dot{\mathbf{u}}$ is piecewise constant, the input $\mathbf{u}$ is simply piecewice linear. Since the input $\mathbf{u}$ can be also used to steer the mobile robot in a desired area, a *second technique* is to embed the singularity avoidance problem in a secondary mobile robot task to be accomplished. In particular we will follows some ideas from redundant robot literature, see, for example, (Chiaverini, 1997) and references therein. Les us define the matrix: $\chi = \mathbf{P}\mathbf{J}(\hat{\mathbf{x}},\mathbf{u}) \in \Re^{3\times 3}$, where the matrices $\mathbf{P}$ and $\mathbf{J}(\hat{\mathbf{x}},\mathbf{u})$ have been introduced in the observer equation (5). In the Prop. 1, fixed $\chi = \mathbf{Q} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}})$, it has been chosen $\mathbf{P} = \chi\,(\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T$ to solve the equation $\chi = \mathbf{P}\mathbf{J}$, assuming $\mathbf{J}$ is full-rank. Let us denote with $\mathbf{P}_i \in (\Re^4)^*$, $i = 1, \ldots 3$ the rows of the matrix $\mathbf{P}$, it is well-known that this matrix solves the problem:

$$\begin{cases} \min\limits_{\mathbf{P}} \sum\limits_{i=1}^{3} \frac{1}{2}\mathbf{P}_i\,\mathbf{P}_i^T \\ \chi = \mathbf{P}\mathbf{J}(\hat{\mathbf{x}},\mathbf{u}) \quad . \end{cases} \quad (15)$$

Instead of minimizing the norm of each row, we can choose to minimize the norm of the difference between $\mathbf{P}_i$ and a suitable $\mathbf{P}_i^*$, to be chosen later. We consider the constrained minimization problem:

$$\begin{cases} \min\limits_{\mathbf{P}} \sum\limits_{i=1}^{3} \frac{1}{2}(\mathbf{P}_i - \mathbf{P}_i^*)\,(\mathbf{P}_i - \mathbf{P}_i^*)^T \\ \chi = \mathbf{P}\mathbf{J}(\hat{\mathbf{x}},\mathbf{u}) \quad . \end{cases} \quad (16)$$

By introducing the Lagrangian multipliers $\lambda_i \in \Re^3$, $i = 1, \ldots, 3$, the problem is solved by considering the function: $L(\mathbf{P},\mathbf{\Lambda}) = \sum_{i=1}^{3}[\frac{1}{2}(\mathbf{P}_i - \mathbf{P}_i^*)\,(\mathbf{P}_i - \mathbf{P}_i^*)^T + (\chi_i - \mathbf{P}_i\,\mathbf{J})\,\lambda_i]$, where $\chi_i \in (\Re^3)^*$, $i = 1, \ldots, 3$, are the rows of the matrix χ and $\mathbf{\Lambda} = [\lambda_1 \; \lambda_2 \; \lambda_3] \in \Re^{3\times 3}$ is the matrix of Lagrangian multipliers. Simple computations show that the conditions:

$$\begin{cases} \dfrac{\partial L(\mathbf{P},\mathbf{\Lambda})}{\partial \mathbf{P}_i^T} = 0 \quad , \\ \dfrac{\partial L(\mathbf{P},\mathbf{\Lambda})}{\partial \lambda_i} = 0 \quad , \; i = 1, \ldots 3 \quad , \end{cases} \quad (17)$$

lead to the solution, if $\mathbf{J}$ is full-rank, expressed in the matrix form: $\mathbf{P} = \chi\,\mathbf{J}^+ + \mathbf{P}^*\,(\mathbf{I} - \mathbf{J}\,\mathbf{J}^+)$, where the matrix $\mathbf{P}^* \in \Re^{3\times 4}$ is the collection of the covectors $\mathbf{P}_i^*$. It is simple to show that Props. 1, and 3 also hold if the matrix $\mathbf{P}$ is chosen in the observer equation (5) as $\mathbf{P} = \chi\,\mathbf{J}^+ + \mathbf{P}^*\,(\mathbf{I} - \mathbf{J}\,\mathbf{J}^+)$ with $\chi = \mathbf{Q} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}})$, since $\mathbf{P}^*\,(\mathbf{I} - \mathbf{J}\,\mathbf{J}^+)\,\mathbf{J} = 0, \forall \mathbf{P}^* \in \Re^{3\times 4}$.

The matrix $\mathbf{P}^*$ can be chosen to obtain a well conditioned EOJ. In fact, let us assume that the input is a function of the actual state estimate, i.e. $\mathbf{u} = \alpha(\hat{\mathbf{x}})$, where $\alpha : \Re^3 \to \Re^2$ is, at least, of class C^1. Given the singularity function $w(\hat{\mathbf{x}},\mathbf{u}) = \sqrt{\det\,(J^T J)}$, the condition (12) (a similar condition is obtained by imposing (14)) yields:

$$\frac{\partial \alpha(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\,[\mathbf{g}(\hat{\mathbf{x}},\mathbf{u}) + \mathbf{P}^*\,(\mu - \hat{\mu})] = k^*\,(\frac{\partial w(\hat{\mathbf{x}},\mathbf{u})}{\partial \mathbf{u}})^T \quad ,(18)$$

where Eq. (5) is used. The above equation is linear in $\mathbf{P}^*$ and imposes two scalar conditions. Hence, a matrix $\mathbf{P}^*$, such that Eq. (12) (or (14)) is satisfied during the vehicle motion, can be computed from (18) by least-square estimation.

5. SIMULATION RESULTS

To investigate the performance of the observer (10), the nonholonomic system in Eq.(4) has been simulated in Matlab environment. In the reported trials, the markers have the following coordinates: $(\bar{\mu}_1 \; \bar{\zeta}_1) = (0, \; 0)$, and $(\bar{\mu}_2 \; \bar{\zeta}_2) = (0, \; 1)$, the control parameters are: $\mathbf{Q} = 2$, and $T = 0.001$. Moreover, we assumed the presence of output measurements noise $\delta\mathbf{y}$, which is band-limited white with power 0.001. In the first simulation, constant input has been applied, i.e. $\mathbf{u} = [1 \; 1]^T$, the initial conditions of the real and estimated state are respectively $\mathbf{x} = [1, \; 1, \; \frac{\pi}{4}]^T$, and $\hat{\mathbf{x}} = [2, \; 0, \; \frac{\pi}{8}]^T$. The results are shown in fig. 1. The simulation indicates the observer stability and tracking of the real state. In fig. 1.d) the singularity index of the EOJ matrix is depicted. When the singularity index becomes suddenly small, numerical problems affect the state tracking, as it happens around the time 5 seconds in the simulation. In the second trial a time-varying input has been applied, namely

$u_1 = 1$, and $u_2 = \sin(t)$, while the initial real state is the same of the previous case, and the estimated state is $\hat{\mathbf{x}} = [1.5,\ 0.5,\ \frac{\pi}{8}]^T$. The results are shown in fig. 2. Also in this case, when the singularity becomes small (see fig. 2.d)) a meaningful state estimation degradation follows. The noise presence amplifies the bad conditioning of the EOJ matrix, and after 7 seconds, the observation error becomes noticeable. To overcome the EOJ matrix bad conditioning, a singularity-avoidance mobile robot exploration technique is then implemented.

To esemplify the singularity-avoidance strategy (see Sect. 4), we report two trials in which the input has been chosen as $\mathbf{u} = \mathbf{\Sigma}\hat{\mathbf{x}}$, where the matrix $\mathbf{\Sigma} \in \Re^{2\times 3}$ is constant and values:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

in particular the input vector does not depend on the estimated angle $\hat{x}_3$. The matrix $\mathbf{P}^*$ is computed, at each iteration, from Eq. (18) by least-square estimation, such that Eq. (14) is satisfied with $k^* = 1$. In the first simulation the initial conditions of the real and estimated state are respectively $\mathbf{x} = [1,\ 1,\ \frac{\pi}{4}]^T$, and $\hat{\mathbf{x}} = [1.5,\ 0.5,\ \frac{3\,\pi}{8}]^T$. The control parameters are: $\mathbf{Q} = 2$, and $T = 0.001$. The results (shown in fig. 3) indicates that the singularity index remains greater than 10^{-8}, which implies a satisfactory quality of the current state estimate. In the second trial, reported in fig. 4, the previous simulation has been repeated in the presence of output measurements noise. In particular, we assumed that $\delta\mathbf{y}$ is band-limited white noise with power 10^{-5}. The results indicate that the observation error is bounded, while the EOJ matrix is well conditioned.

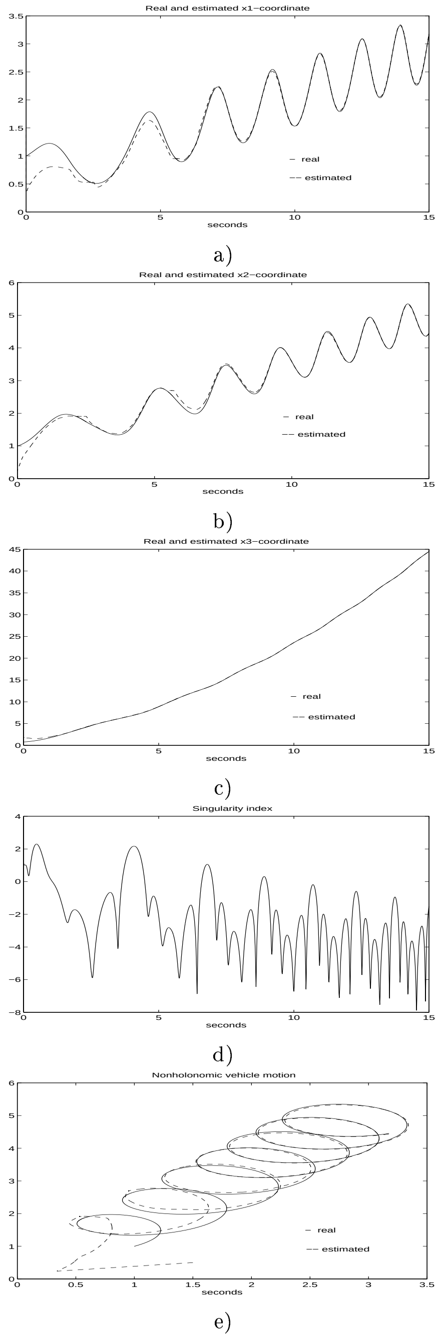

Fig. 3. Nonholonomic vehicle (input avoiding singularities): state estimation without output noise. a) real and estimated x_1 component, b) real and estimated x_2 component, and c) real and estimated x_3 component of the state vector. d) Singularity index of the EOJ matrix: $\log w(\hat{\mathbf{x}}, \mathbf{u})$; and e) vehicle motion in the $\xi - \zeta$ plane.

6. CONCLUSIONS

A nonlinear observer has been proposed for nonholonomic mobile robots, which uses estimated output derivatives via linear high-pass filters, and the concept of EOJ matrix. Uniform observability is a sufficient condition to ensures EOJ generalized inversion, but it is also far to be a necessary condition, since, for our observer design, it is sufficient the local observability of the nonlinear plant under the considered real analytic input, i.e. non singularity of the observability matrix, see (Zeitz, 1987). Future investigations will concern the extension of the proposed approach to more general nonlinear systems and the problem of set-point regulation by using the current state estimate.

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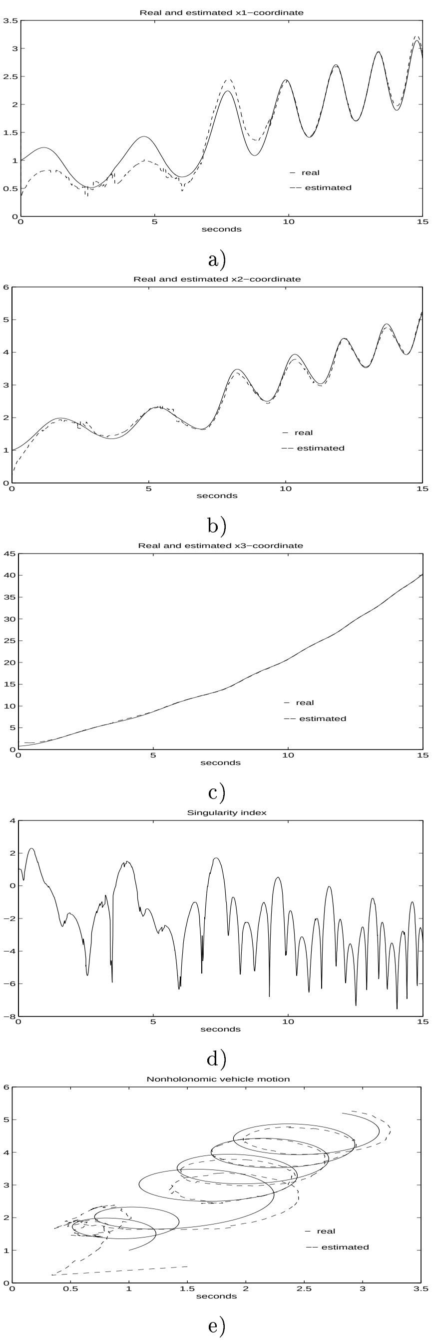

Fig. 4. Nonholonomic vehicle (input avoiding singularities): state estimation in presence of output noise. a) real and estimated x_1 component, b) real and estimated x_2 component, and c) real and estimated x_3 component of the state vector. d) Singularity index of the EOJ matrix: $\log w(\hat{\mathbf{x}}, \mathbf{u})$; and e) vehicle motion in the $\xi - \zeta$ plane.

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