Stabilization of LTI Systems with Quantized State – Quantized Input Static Feedback *

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Abstract. This paper is concerned with the stabilizability problem for discrete–time linear systems subject to a uniform quantization of the control set and to a regular state quantization, both fixed a priori. As it is well known, for quantized systems only weak (practical) stability properties can be achieved. Therefore, we focus on the existence and construction of quantized controllers capable of steering a system to within invariant neighborhoods of the equilibrium.
We first consider uniformly quantized, unbounded input sets for which an increasing family of invariant sets is constructed and quantized controllers realizing invariance are characterized. The family contains a minimal set depending only on the quantization resolution.
The analysis is then extended to cases where the control set is bounded: for any given state–space set of the family above, the minimal diameter of the control set which ensures its invariance is found. The finite control set so determined also guarantees that all the states of the set can be controlled in finite time to within the family's minimal set. It is noteworthy that the same property holds for systems without state quantization: hence, to ensure invariance and attractivity properties, the necessary control set diameter is invariant with state quantization; yet the minimal invariant set is larger. An example is finally reported to illustrate the above results.

1 Introduction

Practical applications of control theory reveal some limits of the continuous models in the description of dynamical systems: limited resources or technical constraints, which finally lead to discrete measurements or to a finite number of possible control actions, are typical situations that must be faced. This is part

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of a broader phenomenon which is referred to as *quantization*.
The example of digitally interconnected systems controlled through finite communication channels (i.e. capable of transmitting only discrete information between the plant and the controller) is usual. Also, many hybrid models (i.e. including the interaction of continuous dynamics and logic) are the result of information quantization.

In the past twenty years the problem of dynamic systems analysis and control synthesis in presence of quantization has developed and is currently growing in interest. It is now consolidated the idea of regarding quantization not as a phenomenon to be neglected and related to the concept of approximation but rather as a useful tool to be studied within proper models (see for instance [1, 2, 5–7, 12–14]).
In the last decade many papers addressed the problem of the stabilization of quantized systems (see [4, 6–12, 15]): in [6] Delchamps clarifies that the classical concept of stability is not significant in this context, hence "practical" stability properties are introduced for quantized systems. Most of the existing literature on stabilization deals with the problem of looking for the quantized resources necessary to achieve a prefixed stability objective.

We are interested in another kind of question that we think is as much important: the stability problem for systems whose quantized control set is *fixed a priori* is studied in [12] where we found a relevant family of invariant sets. In the present paper this analysis is generalized to the case of a prefixed quantization both in the control and in the state space. This is intended to model situations in which, not only the actuators have a discrete or finite set of possible actions, but also measurements provide a limited (i.e. discrete) information on the state of the system. Such analysis is helpful because it allows to decide a priori whether a desired control objective can be achieved by using a *given* technology (actuators, sensors, communication and computational means).

Our work is focused on the stabilization of single–input discrete–time linear systems; we assume that the control space is uniformly quantized and that a reticular quantization is assigned to the state space.
After some preliminaries, Section 3 is dedicated to the construction of a continuous and increasing family of polyhedral invariant sets: the concept of invariance must be reconsidered because, in the quantized state model, although the states evolve according to a deterministic dynamics, the information on them are limited and the controls are selected on the basis of the quantized results of the measurements. A quantized controller (mapping a quantized state into a quantized input set) capable of steering the states in an invariant neighborhood of the equilibrium is constructed. Our analysis does not rely on classical Lyapunov methods but employs direct geometric considerations, which turn out to be less conservative: we characterize the static controllers (i.e. the control laws based only on the current output) realizing the invariance of the sets we have found. The family contains a minimal set depending only on the quantization resolution, its size is increasing with state–space resolution decreases. In Section 4 the analysis is extended to the finite control set case. For any given state–space set

of the family above, the minimal diameter of the control set which ensures its invariance is found. The finite control set so determined also guarantees that all the states of the set can be controlled in finite time to within the family's minimal set. In particular it is constructed a quantized feedback law which both renders invariant a given set of the family and makes the trajectories converge to the family's minimal element: it turns out that the minimal diameter of the control set needed to complete this task is just the same as in the case in which only the inputs are quantized. Hence the state quantization does not influence the bound on the controls necessary to ensure invariance and attractivity properties; yet the minimal invariant set is larger. In Section 5 an example illustrates the presented theoretical results and shows their applicability.

Notation: $Q_n(\Lambda) := \big[-\frac{\Lambda}{2}; \frac{\Lambda}{2} \big]^n$ is the hypercube of edge length Λ, $\lfloor x \rfloor := \max \{z \in \mathbb{Z} \,|\, z \leq x\}$ is the floor function, x^+ is the standard notation for $x(t+1)$, $x_i(t)$ stands for the i^{th} component of the state x at time t, $\|x\|_\infty := \max_{i=1,\ldots,n} \big\{|x_i|\big\}$ and $\mathbf{0} := (0,\ldots,0) \in \mathbb{Z}^n$.

2 Preliminaries

Definition 1. Given a n–tuple $\{w_1,\ldots,w_n\} := \mathcal{W}$ of linearly independent vectors of $\mathbb{R}^n$, for any $(z_1,\ldots,z_n) := \boldsymbol{z} \in \mathbb{Z}^n$, let the *cell* $\mathcal{C}_{\boldsymbol{z}}$ be $\big\{(z_1+a_1)w_1 + \cdots + (z_n+a_n)w_n \,\big|\, a_i \in \big[-\frac{1}{2}, \frac{1}{2} \big) \quad \forall\, i = 1,\ldots,n \big\}$.
Consider the quantized set $\mathcal{S} := \big\{ \sum_{i=1}^n z_i w_i \,\big|\, z_i \in \mathbb{Z} \quad \forall\, i = 1,\ldots,n \big\} \subset \mathbb{R}^n$.
The *reticular quantizer* associated to $\mathcal{W}$ is the function $q_{\mathcal{W}} : \mathbb{R}^n \to \mathcal{S}$ defined as follows: $q_{\mathcal{W}}(x) = z_1 w_1 + \cdots + z_n w_n \Leftrightarrow x \in \mathcal{C}_{\boldsymbol{z}} \;\big(\boldsymbol{z} = (z_1,\ldots,z_n)\big)$.

We deal with a single–input discrete time–invariant linear system subject to a fixed and uniformly quantized control set and to a reticular state quantization, more precisely:

$$\begin{cases} x(t+1) = Ax(t) + bu(t) \\ y(t) = q_{\mathcal{W}}\big(x(t)\big) \\ x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subseteq \epsilon\mathbb{Z} \;\; (\epsilon > 0)\,, \quad y \in \mathcal{S} \subset \mathbb{R}^n \\ A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n. \end{cases} \tag{1}$$

From now on $q_{\mathcal{W}}$ will be simply denoted by q.
A quantizer q models situations where only partial information about the state of the system are available, that is $q(x)$ is known rather than x. More general state–quantizers have been considered in the literature [9]: for the sake of simplicity we restrict to the reticular quantizers, however, as it will be pointed out in Remark 6 at the end of Section 4, the subsequent treatment can be generalized. We associate to system (1) the corresponding system without state quantization, i.e. with q the identity map; it will be denoted by $(A, b, \mathcal{U})$.

We assume that the pair (A, b) is reachable: in this case a change of the coordinates allows us to work with the *controller form* associated to the pair (A, b). Hence, throughout this paper, we will refer to the following hypothesis:

H1) The pair (A, b) is reachable and the system (1) is in controller form. Let $s^n - \alpha_n s^{n-1} - \cdots - \alpha_2 s - \alpha_1$ be the characteristic polynomial of A.

Let us introduce the basic definitions about invariant sets (see also [3]):

Definition 2. The set $D \subseteq \mathbb{R}^n$ is said to be *positively invariant* for a closed–loop system $x^+ = f(x)$ iff $\forall x \in D$, $x^+ \in D$;

Definition 3. The set $D \subseteq \mathbb{R}^n$ is said to be *controlled invariant* for the system $(A, b, \mathcal{U})$ iff $\forall x \in D \; \exists \, u \in \mathcal{U}$ such that $x^+ = Ax + bu \in D$;

Definition 4. The set $D \subseteq \mathbb{R}^n$ is said to be *q–controlled invariant* for system (1) iff $\forall \, x \in D \;\; \exists \, u \in \mathcal{U}$ such that $\forall \, \tilde{x} \in q^{-1}\big(q(x)\big) \cap D$, $\tilde{x}^+ = A\tilde{x} + bu \in D$.

This means that it must be possible to choose a control action, as a function only of the available measurement $q(x)$, such that $x^+ \in D$.

Remark 1. If D is q–controlled invariant for system (1), then it is controlled invariant for the associated system $(A, b, \mathcal{U})$ without state quantization.

The size of the transformed cell $A\,\mathcal{C}_{\boldsymbol{z}}$ along the n^{th} direction is

$$h(A\,\mathcal{C}_{\boldsymbol{z}}) := \sup_{(x', x'') \in \mathcal{C}_{\boldsymbol{z}}^2} \Big\{ \big| (Ax')_n - (Ax'')_n \big| \Big\} ;$$

since $h(A\,\mathcal{C}_{\boldsymbol{z}})$ does not depend on $\boldsymbol{z} \in \mathbb{Z}^n$, we determine it for $\boldsymbol{z} = \mathbf{0}$: the set of the vertices of $\mathcal{C}_{\mathbf{0}}$ is $\mathcal{V} := \big\{ a_1 w_1 + \cdots + a_n w_n \,|\, (a_1, \ldots, a_n) \in \{-\frac{1}{2}, \frac{1}{2}\}^n \big\}$. $\forall v \in \mathcal{V}$, let $h(v) := \big| (Av)_n \big| = \big| \sum_{i=1}^n \alpha_i v_i \big|$. It is easy to see that

$$h(A\,\mathcal{C}_{\mathbf{0}}) = 2 \cdot \max_{v \in \mathcal{V}} h(v) := H .$$

H, which is defined in the controller form coordinates, depends on the coefficients $(\alpha_1, \ldots, \alpha_n)$ of the characteristic polynomial of A and on the quantizer q.

Let $\delta := \sup_{x \in \mathcal{C}_{\mathbf{0}}} \|x\|_\infty$ be the *state–quantizer resolution.*

3 Construction of q–controlled Invariant Sets for $\mathcal{U} = \epsilon\,\mathbb{Z}$

Although invariant sets are very important in control theory, in the current literature few results exist for quantized systems. The input quantization is a severe constraint which often renders unpracticable the classical approaches to the search of controlled invariant sets (see [3]).

In [12] we have found a simple and general technique to construct a family of controlled invariant sets for any uniformly quantized single–input system $(A, b, \mathcal{U})$ such that the pair (A, b) is reachable. The family contains a minimal element which has also good minimality properties with respect to all possible invariant sets.

These results have been derived taking advantage of the controller form coordinates and are summarized in the following

Theorem: If $\mathcal{U} = \epsilon\,\mathbb{Z}$, then $\quad \forall\, \Delta \geq \epsilon\,,\ Q_n(\Delta)$ is controlled invariant.

Owing to Remark 1, it is natural to look for q–controlled invariant sets within the family $\big(Q_n(\Delta)\big)_{\Delta\geq\epsilon}$.

For the unbounded control set case we have the following characterization of the q–controlled invariant hypercubes:

Proposition 1. *Assume that* $\mathcal{U} = \epsilon\,\mathbb{Z}$.

i) If $\frac{\Delta}{2} \geq \delta$, *a necessary condition in order that* $Q_n(\Delta)$ *is* q*–controlled invariant is* $\Delta \geq H$.

ii) A sufficient condition in order that $Q_n(\Delta)$ *is* q*–controlled invariant is* $\Delta \geq H + \epsilon$.

Proof. i) $\mathcal{C}_{\mathbf{0}} \subseteq Q_n(\Delta)$ because $\frac{\Delta}{2} \geq \delta$: hence for the q–controlled invariance of $Q_n(\Delta)$ is necessary that $\exists\, u \in \mathcal{U}$ such that $A\,\mathcal{C}_{\mathbf{0}} + bu \subseteq Q_n(\Delta)$. If $H > \Delta$ then, $\forall\, v \in \mathbb{R}^n,\ A\,\mathcal{C}_{\mathbf{0}} + v \not\subset Q_n(\Delta)$.

ii) Let $x \in Q_n(\Delta)$, $y = q(x) = z_1 w_1 + \cdots + z_n w_n$ is the central point of the cell $\mathcal{C}_{\boldsymbol{z}} \ni x$. The control

$$u(y) := \left(\left\lfloor \frac{-\sum_{i=1}^{n} \alpha_i y_i + \frac{\epsilon}{2}}{\epsilon} \right\rfloor\right)\epsilon$$

realizes the q–controlled invariance of $Q_n(\Delta)$, that is $\forall\, \tilde{x} \in \mathcal{C}_{\boldsymbol{z}} \cap Q_n(\Delta)$, $\tilde{x}^+ = A\tilde{x} + bu(y) \ \in Q_n(\Delta)$, in fact: since $\tilde{x} \in Q_n(\Delta)$ and A is in controller form (so $\tilde{x}^+_j = \tilde{x}_{j+1}\ \forall\, j = 1, \ldots, n-1$), it is sufficient to show that $|\tilde{x}^+_n| \leq \frac{\Delta}{2}$. The central point of the transformed cell $A\,\mathcal{C}_{\boldsymbol{z}}$ is $y^+ = Ay + bu(y)$ and is such that $|y^+_n| \leq \frac{\epsilon}{2}$ (see also [12]), $\tilde{x}^+ \in A\,\mathcal{C}_{\boldsymbol{z}} + bu(y)$, thus $\tilde{x}^+ = y^+ + v$ with $\big|v_n\big| \leq \frac{H}{2}$. Hence $|\tilde{x}^+_n| \leq |y^+_n| + \big|v_n\big| \leq \frac{\epsilon}{2} + \frac{H}{2} \leq \frac{\Delta}{2}$ by the hypothesis. ■

Corollary 1. *Assume that* $\mathcal{U} = \epsilon\,\mathbb{Z}$ *and consider the feedback law* $F:\ \mathbb{R}^n \to \mathcal{U}$

$$F(y) := \left(\left\lfloor \frac{-\sum_{i=1}^{n} \alpha_i y_i + \frac{\epsilon}{2}}{\epsilon} \right\rfloor\right)\epsilon\,.$$

The induced closed–loop dynamics

$$x^+ = Ax + bF\big(q(x)\big) \tag{2}$$

for the state–quantized system (1) is such that all $x \in \mathbb{R}^n$ *are steered into* $Q_n(H+\epsilon)$ *in at most* n *steps and* $Q_n(H+\epsilon)$ *is positively invariant.* □

The function $\big(F \circ q\big):\ \mathbb{R}^n \to \mathcal{U}$ is a quantized state–quantized input version of the so–called dead–beat controller (whereas F is referred to as the quantized input dead–beat controller).

Remark 2. Note that in Proposition 1.*ii* we do not require that $\frac{\Delta}{2} \geq \delta$ $\big($which is equivalent to the existence of a cell $\mathcal{C}_{\boldsymbol{z}} \subseteq Q_n(\Delta)\big)$: when $H + \epsilon < 2\,\delta$, for $\Delta \in [H + \epsilon, 2\,\delta)$ it holds that $\forall x \in Q_n(\Delta)$ the measurement $q(x)$ is not sufficient to guarantee that $x \in Q_n(\Delta)$; in this case Proposition 1.*ii* seems to reduce to a formal assertion. This is not the case: in fact, assume for instance that $H + \epsilon < 2\,\delta$ and that the system evolves according to the closed–loop dynamics (2), then, even if the measurements $q(x)'s$ are not sufficient to show that $x \in Q_n(H+\epsilon)$, it is known a priori that from the n^{th} step on $x \in Q_n(H+\epsilon)$. Thus it is not unrealistic to investigate the q–controlled invariance for hypercubes of edge length $\Delta < 2\,\delta$.
In Section 5 we will give an example where this phenomenon occurs.

We conclude this section with the characterization of the quantized controllers which make $Q_n(\Delta)$ positively invariant.

Assume that $\mathcal{U} = \epsilon\,\mathbb{Z}$, fix $\Delta \geq H + \epsilon$ and consider $Q_n(\Delta)$. Let $\mathcal{S}_{[\Delta]} := \mathcal{I}m\big(q_{|Q_n(\Delta)}\big) \subset \mathcal{S}$. $\forall\, y \in \mathcal{S}_{[\Delta]}$ let $\mathcal{C}_{\boldsymbol{z}(y)}$ be the cell containing y and set $H^y := \sum_{i=1}^n \alpha_i y_i + \frac{H}{2}$ and $H_y := \sum_{i=1}^n \alpha_i y_i - \frac{H}{2}$ which respectively denote the sup and the inf of the n^{th} component of the points of the transformed cell $A\,\mathcal{C}_{\boldsymbol{z}(y)}$. The set

$$\mathcal{U}_{[\Delta,y]} := \Big\{ u \in \mathcal{U} \;\Big|\; \forall\, x \in \mathcal{C}_{\boldsymbol{z}(y)} \cap Q_n(\Delta)\,,\; x^+ \in Q_n(\Delta) \Big\}$$

consists of the controls which realize the q–controlled invariance of $Q_n(\Delta)$ when the measurement $y = q(x)$ is available. By arguments similar to those used to prove Proposition 1.*ii*, it can be shown that $\mathcal{U}_{[\Delta,y]} \supseteq \mathcal{U}^*_{[\Delta,y]}$, where

$$\mathcal{U}^*_{[\Delta,y]} := \Big\{ z\,\epsilon \,\Big|\, z \in \mathbb{Z} \;\text{ and }\; -\Big\lfloor \tfrac{1}{\epsilon}\big(\tfrac{\Delta}{2} + H_y\big) \Big\rfloor \leq z \leq \Big\lfloor \tfrac{1}{\epsilon}\big(\tfrac{\Delta}{2} - H^y\big) \Big\rfloor \Big\};$$

if moreover $\mathcal{C}_{\boldsymbol{z}(y)} \subseteq Q_n(\Delta)$, then $\mathcal{U}^*_{[\Delta,y]} = \mathcal{U}_{[\Delta,y]}$.

Using the definition of the floor function we calculate $\#\,\mathcal{U}^*_{[\Delta,y]} = \frac{\Delta - H}{\epsilon} - \theta$ with $\theta \in [-1, 1)$; in particular, for $\Delta = H + \epsilon$, $0 < \#\,\mathcal{U}^*_{[H+\epsilon,y]} \leq 2$.

Remark 3. Since $\forall\, y \in \mathcal{S}_{[\Delta]}$, $\#\,\mathcal{U}_{[\Delta,y]} < +\infty$ and also $\#\,\mathcal{S}_{[\Delta]} < +\infty$, then there exists a finite number of static quantized controllers defined in $Q_n(\Delta)$ which make it positively invariant, that is

$$\#\,\Big\{ \Phi :\; \mathcal{S}_{[\Delta]} \to \mathcal{U} \;\Big|\; \forall\, x \in Q_n(\Delta)\,,\; x^+ = Ax + b\,\Phi\big(q(x)\big) \in Q_n(\Delta) \Big\} < +\infty\,.$$

4 Finite Control Set

We now analyze the q–controlled invariance of the hypercubes $Q_n(\Delta)$'s (with $\Delta \geq H + \epsilon$) in the finite control set case. We consider input sets of the type

$$\mathcal{U}_k := \{-k\epsilon, \ldots, 0, \ldots, +k\epsilon\}$$

and, for a given $\Delta \geq H + \epsilon$, we find the condition on k ensuring the q–controlled invariance of $Q_n(\Delta)$.

We restrict our analysis to systems such that $\sum_{i=1}^{n} |\alpha_i| \geq 1$ which are indeed the interesting ones: in fact in the other case, not only the system is stable, but also $u \equiv 0$ is sufficient for the invariance of any hypercube and for the convergence of the trajectories to the equilibrium. Note that when $\sum_{i=1}^{n} |\alpha_i| \geq 1$, it holds that $\|A\|_\infty = \sum_{i=1}^{n} |\alpha_i|$.

Consider the system $(A, b, \mathcal{U}_k)$ associated to system (1): in [12] we have proved that

Theorem: $Q_n(\Delta)$ is controlled invariant if and only if

$$k \geq -\Big\lfloor \frac{1}{2} \frac{\Delta}{\epsilon} \big(1 - \sum_{i=1}^{n} |\alpha_i| \big) \Big\rfloor := \mathcal{K}.$$

By Remark 1 it follows that for the q–controlled invariance of $Q_n(\Delta)$ it is necessary that $k \geq \mathcal{K}$. In next Proposition 2, we construct an explicit quantized feedback law taking values in $\mathcal{U}_\mathcal{K}$ and rendering $Q_n(\Delta)$ positively invariant. Hence the condition $k \geq \mathcal{K}$ is also sufficient: this means that, even if the state space is quantized, it is not necessary to have more control resources to ensure invariance properties.

Proposition 2. *Assume that* $\sum_{i=1}^{n} |\alpha_i| \geq 1$; *let* $\Delta \geq H + \epsilon$ *and* $k := -\Big\lfloor \frac{1}{2} \frac{\Delta}{\epsilon} \big(1 - \sum_{i=1}^{n} |\alpha_i| \big) \Big\rfloor$. *Consider the feedback law* $\tilde{F} : \mathbb{R}^n \to \mathcal{U}_k$ *defined by*

$$\tilde{F}(y) := \begin{cases} -(k\epsilon) & \text{if} \quad \sum_{i=1}^{n} \alpha_i y_i - k\epsilon \geq \frac{\epsilon}{2}, \\ +(k\epsilon) & \text{if} \quad \sum_{i=1}^{n} \alpha_i y_i + k\epsilon \leq -\frac{\epsilon}{2}, \\ z\epsilon & \text{with} \quad z = \Big\lfloor \frac{-\sum_{i=1}^{n} \alpha_i y_i + \frac{\epsilon}{2}}{\epsilon} \Big\rfloor \quad \text{otherwise.} \end{cases} \tag{3}$$

Then $\forall \gamma \in [H + \epsilon, \Delta]$, $Q_n(\gamma)$ *is positively invariant for*

$$x^+ = Ax + b\tilde{F}\big(q(x)\big). \tag{4}$$

Proof. Let $\Xi := \big\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i x_i - k\epsilon > \frac{\epsilon}{2} \big\} \cup \big\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i x_i + k\epsilon < -\frac{\epsilon}{2} \big\}$ be the region where the quantized input dead–beat controller saturates. Note that $\forall x \notin \Xi$, $v := Ax + b\tilde{F}(x)$ is such that $\big| v_n \big| \leq \frac{\epsilon}{2}$.
Since A is in controller form, for $x \in Q_n(\gamma)$ it is sufficient to analyze x_n^+; we divide the analysis in three cases.
I) If $q(x) \notin \Xi$ then, using the same arguments used to prove Proposition $1.ii$, $|x_n^+| \leq \frac{H+\epsilon}{2}$.
II) If $q(x) \in \Xi$ and $x \notin \Xi$ then, with $y = q(x)$, $y^+ = Ay + b\tilde{F}(y)$ is such that $|y_n^+| > \frac{\epsilon}{2}$. Suppose that $\sum_{i=1}^{n} \alpha_i y_i > 0$, then $y_n^+ = \sum_{i=1}^{n} \alpha_i y_i - k\epsilon > \frac{\epsilon}{2}$. By Equation (4), $x_n^+ = \sum_{i=1}^{n} \alpha_i x_i - k\epsilon \leq \sum_{i=1}^{n} \alpha_i x_i + \tilde{F}(x) \leq \frac{\epsilon}{2}$ because $x \notin \Xi$; moreover $x_n^+ \geq y_n^+ - \frac{H}{2} > \frac{\epsilon}{2} - \frac{H}{2} > -\frac{H+\epsilon}{2}$: thus $|x_n^+| \leq \frac{H+\epsilon}{2}$.
The case $\sum_{i=1}^{n} \alpha_i y_i < 0$ is similar.

III) If $q(x) \in \Xi$ and $x \in \Xi$ then $x^+ = Ax + b\tilde{F}\big(q(x)\big) = Ax + b\tilde{F}(x)$. If $\sum_{i=1}^n \alpha_i x_i > 0$, since $x \in \Xi$, then $\sum_{i=1}^n \alpha_i x_i - k\epsilon > \frac{\epsilon}{2} > 0$. Hence $|x_n^+| = \sum_{i=1}^n \alpha_i x_i - k\epsilon \le \sum_{i=1}^n |\alpha_i|\,|x_i| - k\epsilon \le \|x\|_\infty \cdot \sum_{i=1}^n |\alpha_i| - k\epsilon$: in this case the proof of the statement is achieved by showing that $\|x\|_\infty \cdot \sum_{i=1}^n |\alpha_i| - k\epsilon \le \|x\|_\infty$. By the definition of k it holds that $k\epsilon \ge \frac{\Delta}{2}\big(\sum_{i=1}^n |\alpha_i| - 1\big) \ge \|x\|_\infty \big(\sum_{i=1}^n |\alpha_i| - 1\big)$ because $\frac{\Delta}{2} \ge \|x\|_\infty$ and $\sum_{i=1}^n |\alpha_i| \ge 1$. Thus $k\epsilon \ge \|x\|_\infty \big(\sum_{i=1}^n |\alpha_i| - 1\big)$ which is what we wanted to show.
The case $\sum_{i=1}^n \alpha_i x_i < 0$ is similar. ■

Corollary 2. *Consider the system (1) and assume that $\mathcal{U} = \mathcal{U}_k$, let $\Delta \ge H + \epsilon$. $Q_n(\Delta)$ is q–controlled invariant if and only if*

$$k \ge -\Big\lfloor \frac{1}{2}\frac{\Delta}{\epsilon}\Big(1 - \sum_{i=1}^n |\alpha_i|\,\Big)\Big\rfloor .$$

In particular, for $\Delta \ge H + \epsilon$, $Q_n(\Delta)$ is q–controlled invariant for system (1) if and only if it is controlled invariant for the associated system $(A, b, \mathcal{U}_k)$ without state quantization. □

Note that the closed–loop dynamics in Equation (4) is such that if $\frac{H+\epsilon}{2} \le \|x\|_\infty \le \frac{\Delta}{2}$, then $\|x^+\|_\infty \le \|x\|_\infty$: in next Proposition 3 we will show that a mild supplementary hypothesis is sufficient to ensure that any trajectory starting from $x(0) \in Q_n(\Delta)$ enters $Q_n(H + \epsilon)$ in a finite number of steps.

Proposition 3. *Assume that $\sum_{i=1}^n |\alpha_i| \ge 1$; let $\Delta \ge H + \epsilon$ and $k := -\Big\lfloor \frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^n |\alpha_i|\,\big)\Big\rfloor$. If $\frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^n |\alpha_i|\,\big) \notin \mathbb{Z}$, then the closed–loop dynamics*

$$x^+ = Ax + b\tilde{F}\big(q(x)\big),$$

induced by the feedback law $\tilde{F} : \mathbb{R}^n \to \mathcal{U}_k$ defined in Equation (3), is such that $Q_n(\Delta)$ is positively invariant, all $x(0) \in Q_n(\Delta)$ are steered into $Q_n(H + \epsilon)$ in a finite number of steps and $Q_n(H + \epsilon)$ is positively invariant.
For $x(0) \in Q_n(\Delta) \setminus Q_n(H+\epsilon)$, an upper bound on the number of steps necessary to enter $Q_n(H + \epsilon)$ is given by

$$\mathcal{B} := -\,n \cdot \Bigg\lfloor \frac{1}{\varphi}\Bigg(\frac{H+\epsilon}{2} - \Big(\big\|q\big(x(0)\big)\big\|_\infty + \delta\Big)\Bigg)\Bigg\rfloor ,$$

where $\varphi := k\epsilon - \frac{\Delta}{2}\big(\sum_{i=1}^n |\alpha_i| - 1\big)$ and δ is the state–quantizer resolution.

Proof. The positive invariance of $Q_n(\Delta)$ and $Q_n(H + \epsilon)$ has been proved in Proposition 2. From $\frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^n |\alpha_i|\,\big) \notin \mathbb{Z}$ it follows immediately that $\varphi > 0$. We claim that $\forall\, x \in Q_n(\Delta) \setminus Q_n(H+\epsilon)$, x^+ is such that $|x_n^+| \le \frac{H+\epsilon}{2}$ or $|x_n^+| \le \|x\|_\infty - \varphi$. The claim implies the thesis, in fact: since A is in controller form, after n steps it holds that $\big|x_j(n)\big| \le \max\big\{\|x(0)\|_\infty - \varphi\,;\, \frac{H+\epsilon}{2}\big\}$ $\forall\, j = 1, \ldots, n$; thus

$\|x(n)\|_\infty \leq \max\left\{\|x(0)\|_\infty - \varphi\,;\, \frac{H+\epsilon}{2}\right\}$. Since φ is a strictly positive constant, the thesis follows.
The bound on the number of steps necessary to enter $Q_n(H+\epsilon)$ is obtained by looking for the smallest $m \in n\,\mathbb{N}$ such that $\|x(0)\|_\infty - \frac{m}{n}\,\varphi \leq \frac{H+\epsilon}{2}$: by simple calculations we get $m = -\,n \cdot \left\lfloor \frac{1}{\varphi}\left(\frac{H+\epsilon}{2} - \|x(0)\|_\infty\right)\right\rfloor \leq \mathcal{B}$ because $\|x(0)\|_\infty \leq \left\|q\big(x(0)\big)\right\|_\infty + \delta$.
Let us prove the claim: if $x \notin \Xi$ or $q(x) \notin \Xi$ then in the proof of Proposition 2 we have shown that $|x_n^+| \leq \frac{H+\epsilon}{2}$. If $x \in \Xi$ and $q(x) \in \Xi$, for $\sum_{i=1}^n \alpha_i x_i > 0$ it holds that $|x_n^+| \leq \|x\|_\infty \cdot \sum_{i=1}^n |\alpha_i| - k\epsilon$, as shown in part III of the proof of Proposition 2; by the definition of φ, $\|x\|_\infty \cdot \sum_{i=1}^n |\alpha_i| - k\epsilon = \|x\|_\infty \cdot \sum_{i=1}^n |\alpha_i| - \varphi - \frac{\Delta}{2}\big(\sum_{i=1}^n |\alpha_i| - 1\big) \leq \|x\|_\infty - \varphi$.
The case $\sum_{i=1}^n \alpha_i x_i < 0$ is similar. ■

Note that if $\frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^n |\alpha_i|\ \big) \in \mathbb{Z}$ and $\alpha_i \geq 0 \quad \forall\, i = 1, \ldots, n$ $\big($with $\sum_{i=1}^n \alpha_i \geq 1\big)$, then $x = \big(\frac{\Delta}{2}, \cdots, \frac{\Delta}{2}\big)$ is such that $\exists!\ u \in \mathcal{U}_k$ ensuring that $x^+ \in Q_n(\Delta)$; in this case $x^+ = x$, therefore x is not attracted by $Q_n(H+\epsilon)$. Anyway, if the condition $\frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^n |\alpha_i|\ \big) \notin \mathbb{Z}$ does not hold, then one more level of controls $\big($i.e. $\mathcal{U} = \mathcal{U}_{k+1}\big)$ is sufficient to guarantee the attractivity of $Q_n(H+\epsilon)$.

Remark 4. Exactly as in the case in which only the input are quantized (see [12]), it holds that the minimal diameter of the control set (the saturation level) needed to ensure the invariance of $Q_n(\Delta)$ is also sufficient to guarantee that all the states of $Q_n(\Delta)$ are initial points of trajectories which lie within $Q_n(\Delta)$ and are attracted towards $Q_n(H+\epsilon)$. This property can be profitably exploited to reduce the amount of resources necessary to complete the stabilization task. For instance, when the dead–beat controller is not saturated, the maximal value that it takes within $Q_n(\Delta)$ is approximately $-\left\lfloor \frac{1}{2}\big(1 - \frac{\Delta}{\epsilon}\sum_{i=1}^n |\alpha_i|\ \big)\right\rfloor$: hence the optimal saturation makes possible to save about $\frac{1}{2}\frac{\Delta}{\epsilon}$ levels.

Remark 5. Even though the bound $\mathcal{B}$ on the number of steps necessary to enter the final set can be updated at each step, it is a very conservative estimate.
Basically there are three ways to know that the state has reached the final set:

A- $q(x)$ corresponds to a cell $\mathcal{C}_{\boldsymbol{z}} \subseteq Q_n(H+\epsilon)$;
B- if $q(x) \notin \Xi$ for n consecutive steps then, by the part I of the proof of Proposition 2 and the controller form of A, we deduce that at the successive step $x \in Q_n(H+\epsilon)$;
C- the use of the bound $\mathcal{B}$.

The third case must be considered just as a parachute in case that A and B fail.

Remark 6 (Beyond reticular quantization). The only relevant information about the state–quantizer $q_{\mathcal{W}}$ which have been involved in the foregoing results are the quantities H and δ: this enables us to apply the presented techniques to more general state–quantizers and to get similar results.

5 Example

Consider the system

$$x^+ = \begin{pmatrix} 0 & 1 \\ \frac{5}{4} & \frac{1}{4} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u\,;$$

suppose that $\mathcal{U} \subseteq \frac{1}{4}\mathbb{Z}$, thus $\epsilon = \frac{1}{4}$, and that the reticular state quantization is associated to

$$\mathcal{W} = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right\} :$$

in this case $\delta = 2$ and $H = \frac{7}{2}$. It is worth noting that since $H + \epsilon = \frac{15}{4} < 2 \cdot \delta = 4$, the set $Q_2(H+\epsilon)$ does not contain any cell of the state quantization (see the figure), in particular the criterion `A` of Remark 5 can not be used in this case.

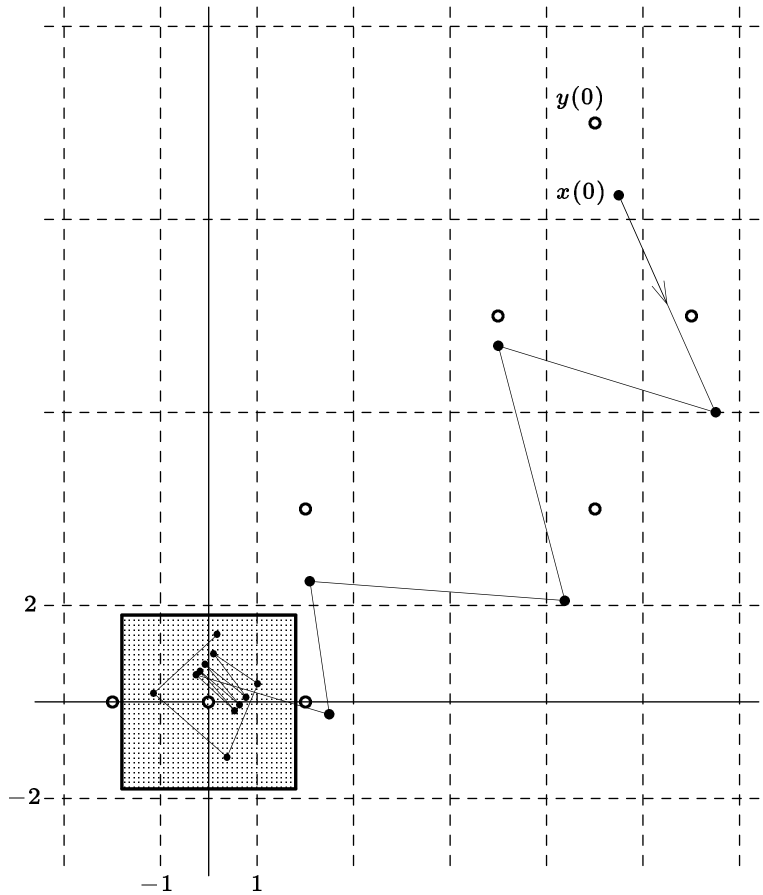
Let us suppose that at time 0 the quantized result of the measurement of the state $x(0)$ is $y(0) = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$: with $\Delta = 28.1$ we ensure that $x(0) \in Q_2(\Delta)$. According to Proposition 3, it holds that $k = 29$ and $\varphi = \frac{9}{40}$; let $\mathcal{U} = \mathcal{U}_{29} \subset \frac{1}{4}\mathbb{Z}$ and implement the feedback law defined in Equation (3). The observations of the evolution of the system are summarized in the following table:

Step	0	1	2	3	4	5	6	7	8
y	$\binom{8}{12}$	$\binom{10}{8}$	$\binom{6}{8}$	$\binom{8}{4}$	$\binom{2}{4}$	$\binom{2}{0}$	$\binom{0}{0}$	$\binom{0}{0}$	$\binom{0}{0}$
$u(y)$	$-\frac{29}{4}$	$-\frac{29}{4}$	$-\frac{29}{4}$	$-\frac{29}{4}$	$-\frac{14}{4}$	$-\frac{10}{4}$	0	0	0
$\mathcal{B}$	108	90	74	74	38	20			
Step	9	10	11	12	13	14	15	16	17
y	$\binom{0}{0}$	$\binom{0}{0}$	$\binom{0}{0}$	$\binom{0}{0}$	$\binom{2}{0}$	$\binom{0}{0}$	$\binom{-2}{0}$	$\binom{0}{0}$	
$u(y)$	0	0	0	0	$-\frac{10}{4}$	0	$\frac{10}{4}$	0	
$\mathcal{B}$									

Since at the 4^{th} and 5^{th} step the controller does not saturate then, using the criterion `B` of Remark 5, we deduce that from the 6^{th} step on the state x is confined within $Q_2(H+\epsilon)$ (hence the computation of $\mathcal{B}$ has been stopped). We also note that, in spite of the state quantization, just three control values are sufficient to make $Q_2(H+\epsilon)$ invariant.

According to Remark 4, the feedback law defined in Equation (3) makes possible to save approximately $\frac{1}{2}\frac{\Delta}{\epsilon} \simeq 56$ levels in the control set.

The observed behavior is generated by $x(0) = \begin{pmatrix} 17/2 \\ 21/2 \end{pmatrix}$: the following figure shows the real evolution of the state (denoted with black circles "$\bullet$"), the white circles "$\circ$" are the output $q(x)$'s, that is the central points of the cells visited by the state, the shaded square is $Q_2(H+\epsilon)$.



Conclusions

In this paper we have addressed the stabilization analysis for discrete–time linear systems subject to a fixed uniform quantization both in the control and in the state space. We have focused on the study of invariant neighborhoods of the equilibrium and provided quantized controllers steering the system into such sets (i.e. realizing attractivity). Several open problems remain in this field, among which notably is the extension to dynamic feedback of quantized state information, and quantized output feedback. More generally, the combination of quantization with limited communication bandwidths is a most important and challenging area to which further work will be devoted.

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