Dynamic Analysis of Mobility and Graspability of General Manipulation Systems

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***Abstract*— We present a geometric approach to the dynamic analysis of manipulation systems of a rather general class, including some important types of manipulators as, e.g., cooperating, super-articulated, and whole-arm manipulators. The focus is in particular on simple industry-oriented devices, for which a minimalistic design approach requires a clear understanding of mobility and graspability properties in the presence of kinematic defectivity. The paper discusses the dynamics of these systems, and considers how their structural properties (in the classical system-theoretic sense, i.e., stability, controllability, observability, etc.) are related to frequently used concepts in robotics such as "redundancy," "graspability," "mobility," and "indeterminacy." Less common or novel concepts, such as those of "defectivity," "hyperstaticity," and "dynamic graspability," are elicited and/or enlightened by this study. Some important practical consequences of the limited control possibilities of defective systems are thus put into evidence. Finally, a standard form of the dynamics of general manipulation systems is provided as a compact and readable synopsis of the dynamic structure. The form is a valuable tool for synthesizing dynamic controllers for such systems, especially suited to geometric control design methods.**

I. Introduction

ONE of the main avenues of research in robotic manipulation is the development of robot systems whose mechanical structure is more complex than that of conventional serial-linkage arms. One instance of this is the coordinated use of multiple fingers in a robot hand, or, similarly, of multiple arms in cooperating tasks and of multiple legs in a vehicle for locomotion. Unilateral contact phenomena between different members of the system are also often encountered. Thus, in a robot hand, each finger acts on the manipulated object through a passive (not directly actuated) "joint" consisting of a mechanical contact, which is subject to inequality constraints on the direction of forces, and to kinematic constraints on rolling and sliding motions. Passive joints may also be present in the mechanism on purpose as, e.g., happens in the class of super-articulated systems studied by Seto and Baillieul [33].

A recent innovation consists in the exploitation of all links of the limbs to manipulate objects, rather than using only their end-effectors (whole-arm manipulation, see [32]). A peculiarity of whole-limb systems is that inner links generally have fewer degrees-of-freedom than necessary to achieve arbitrary configurations in their operational space, i.e., are kinematically defective. Similar considerations apply to other manipulation structures, such as bracing systems (see e.g., [8]) and parallel manipulators, pioneered by Stewart [35].

This paper reports on an endeavour at attacking the analysis of such diverse styles of manipulation uniformly.

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A. Previous Work

The origin of the analysis of general manipulation systems can be traced back to three Ph.D. dissertations studying multifingered hands. Salisbury [30] set up the foundations of a linear algebraic approach to the problem. Kerr [17] considered a wide spectrum of manipulation systems, including explicit reference to their defectivity, indeterminacy, redundancy, etc. Trinkle [36] studied the mechanics of enveloping grasps, and provided planning strategies for such systems. Most of these works were based on quasistatic assumptions.

While literature on nondefective manipulation systems has since then grown extensively, less work has been devoted to the general case, almost always restricting to quasistatic assumptions. Pettinato and Stephanou [26] described a tentacle-based manipulator and analyzed its manipulability and contact stability. Mirza and Orin [21] described a multiple arm manipulation system (DIGITS), and discussed the improved robustness of power grasping. Hunt *et al*. [15] considered the kinematics of composite serial/in-parallel manipulators, while Waldron and Hunt [38] discussed the series/parallel duality from the kineto-static viewpoint. Bicchi [6] made explicit the limitations to the arbitrariness in distributing manipulation forces among cooperating limbs due to the presence of defective elements (a problem which was previously noticed by Kerr [17] and Trinkle [37]). Bicchi, Melchiorri, and Balluchi [7] studied the rigid-body kinematics of WAM systems and discussed their manipulability.

Dynamic analysis of manipulation systems has attracted relatively less attention so far. This is partly justified by the fact that most cooperative manipulation tasks are slow enough to render dynamic effects negligible (notable exception to this are discussed in [10], [24]. Although dynamics may not play a dominant role in the performance of slow cooperative manipulation tasks, it is true that only a full dynamical model can explain and clarify the structural properties of complex manipulation systems. Thus, dynamic manipulation has been considered to investigate grasp stability [3], [13], [14], [22], [25], [37], to study the dualities between series and parallel manipulators [40]; and to address cooperative manipulability [9], [19].

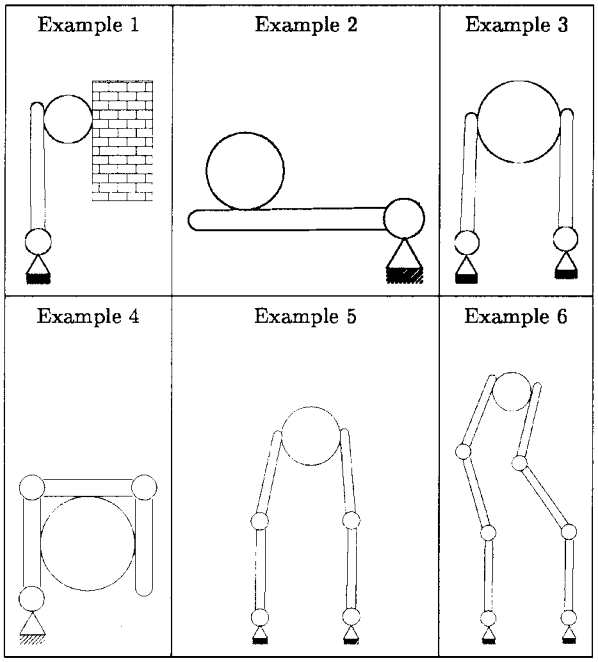

Fig. 1. Six simple examples of robotic manipulators.

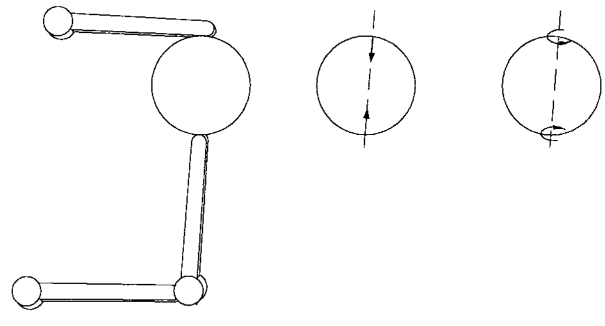

Fig. 2. Kerr and Roth's example (a). Any linear combination of forces as in (b) and (c) is internal, but only forces as in (b) are controllable internal.

B. Contributions

The purpose of this paper is to contribute to building a theory for a class of robotic manipulation systems that is general enough to include most practical manipulation systems. The paper is an extension of previous work on coordinated manipulation system analysis [17], [30], [37], to include dynamics and structural properties other than stability. In particular, restrictions to controllability/observability entailed by kinematic defectivity are enlightened. As an instance of this problem, consider the example in Fig. 2, redrawn from [18]. In that important early paper, authors were concerned with choosing optimal internal forces to grasp the object, and did so by choosing (by linear programming methods) a combination of all possible internal forces/moments, including opposite torques about the line through the contacts. On the other hand, it is intuitive to observe that there is no possibility of actually applying such torsion to the object by the depicted mechanism. The analysis presented in this paper allows to thoroughly describe and understand this situation (see Section IV-B), and accordingly restrict the search for optimal grasping forces within the proper sets.

To solve this and other problems, the paper uses classical system-theoretic concepts of dynamic system analysis (stability, controllability, observability etc.) and relates them to frequently used concepts in robotics such as "redundancy," "graspability," "mobility," "indeterminacy," "defectivity," and "hyperstaticity." Less common concepts such as those of "controllable internal forces" and "dynamically internal forces," are introduced, and their important practical implications are enlightened by this study.

Furthermore, a finely decoupled standard form of the dynamics of general manipulation systems is provided as a compact and readable synopsis of the dynamic structure. The main application of this result is in the synthesis of controllers for solving some important problems (such as the disturbance-decoupling and the noninteractive control problem, see e.g., [29]) through the usage of geometric control design techniques [2], [39].

To the best of our knowledge, no such systematic dynamic analysis exists to date, due in part to the novelty of the subject, and in part to the apparent intractability of the computations in the full nonlinear case. Our approach to the problem is based on the use of geometric system-theoretical tools on a linearized model. Although only local results can be inferred by this method, linearized dynamic analysis certainly represents a significant advancement with respect to quasistatic studies (a discussion on this point is reported in Section VII).

Throughout the paper, we refer to a most simple set of examples, reported in Fig. 1. The set contains examples of manipulators representing an idealization of more complex devices that may be encountered in practice, and also some "pathological" cases of no practical interest besides their illustrative purpose. Relevant numerical data for such examples are reported in Appendix C.

II. Dynamic Model

The class of "general manipulation systems" this paper is concerned with is comprised of mechanisms with any number of limbs (open kinematic chains), of joints (prismatic, rotoidal, spherical, etc.) and of contacts (hard and soft finger, complete-constraint, etc.) between a reference member called "object" and links in any position in the limb chains. This includes in particular defective and hyperstatic systems, whose treatment is seldom considered in the literature.

As a paradigm for general manipulation systems, we refer to the case of a multifingered hand manipulating an object through contacts on its finger parts and palm (see Fig. 3). Let $\boldsymbol{q} \in \mathbb{R}^q$ denote the vector of joint positions, and let $\boldsymbol{u} \in \mathbb{R}^d$ be the vector locally describing the position and orientation of a frame attached to the object [$d = 3$ for planar systems, $d = 6$ for systems in three-dimensional (3-D) space]. Correspondingly, let $\tau \in \mathbb{R}^q$ be the vector of forces and torques of the joint actuators, and $\boldsymbol{w} \in \mathbb{R}^{\boldsymbol{d}}$ the vector of forces and torques resultant from actions applied directly at the object.

Hand and object dynamics are linked through n rigid-body unilateral contact constraints that, according to Appendix A, can be written in terms of the grasp matrix $\boldsymbol{G}$ and hand

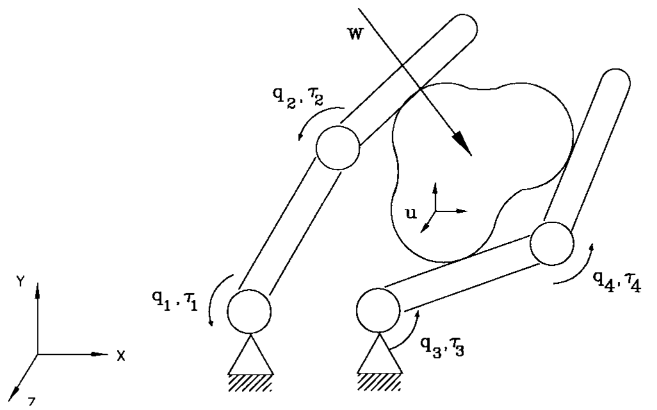

Fig. 3. Introducing some notation for a general manipulation system: $\boldsymbol{q} \in \mathbb{R}^4, \tau \in \mathbb{R}^4, \boldsymbol{u} \in \mathbb{R}^6, \boldsymbol{w} \in \mathbb{R}^6$.

Jacobian $\boldsymbol{J}$ as

$$[\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}]\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{u}}\end{bmatrix} = \mathbf{0}. \tag{1}$$

Notice that the number t of constraint equations depends on the contact models used to describe the n contact interactions [31], [11]).

By introducing a t-dimensional vector t of Lagrangian multipliers and by differentiating (1), rigid-body dynamics equations are obtained as

$$\begin{aligned}\boldsymbol{M}_h(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{Q_h}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{J}^{\mathrm{T}}\mathrm{t} &= \tau\\ \boldsymbol{M}_o(\boldsymbol{u})\ddot{\boldsymbol{u}} + \boldsymbol{Q_o}(\boldsymbol{u},\dot{\boldsymbol{u}}) - \boldsymbol{G}\mathrm{t} &= \boldsymbol{w}\\ \boldsymbol{J}\ddot{\boldsymbol{q}} - \boldsymbol{G}^{\mathrm{T}}\ddot{\boldsymbol{u}} + \boldsymbol{Q_c}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{u},\dot{\boldsymbol{u}}) &= \mathbf{0}\end{aligned} \tag{2}$$

where $\boldsymbol{Q_c} = (\partial \boldsymbol{J}\dot{\boldsymbol{q}}/\partial \boldsymbol{q})\dot{\boldsymbol{q}} - (\partial \boldsymbol{G}^{\mathrm{T}}\dot{\boldsymbol{u}}/\partial \boldsymbol{u})\dot{\boldsymbol{u}}$ and $\boldsymbol{M}_h(\cdot)$ and $\boldsymbol{M}_o(\cdot)$ are symmetric and positive definite inertia matrices and $\boldsymbol{Q}_h(\cdot,\cdot)$ and $\boldsymbol{Q}_o(\cdot,\cdot)$ are terms including velocity-dependent and gravity forces of the hand and of the object, respectively.

From (1) and (2), it appears the central role played by the Jacobian and grasp matrices. With respect to their structure, some relevant characteristics of the manipulation system are introduced.

Definition 1: A manipulation system is said *redundant* if $\ker \boldsymbol{J} \neq \mathbf{0}$.

In a redundant system, there exist "internal" motions of the fingers alone that do not violate the contact constraint (1). For a given configuration of the grasped object, an infinity of neighboring hand configurations are feasible. The part of Fig. 4, labeled $\mathcal{N}_J$, illustrates a redundant motion.

Definition 2: A manipulation system is said kinematically *indeterminate* if $\ker \boldsymbol{G}^{\mathrm{T}} \neq \mathbf{0}$.

In an indeterminate grasp, there exist motions of the object alone that do not violate (1). Indeterminacy implies that the object is not firmly grasped, because for a given configuration of the hand, an infinity of neighboring configurations of the object are feasible. The part of Fig. 4, labeled $\mathcal{N}_{G^{\mathrm{T}}}$, illustrates an indeterminate motion.

Definition 3: A manipulation system is said *graspable* if $\ker \boldsymbol{G} \neq \mathbf{0}$.

Graspable systems exhibit self-balanced constraint forces t, resulting in zero net force on the object, cf. (2). In the literature, forces belonging to the nullspace of $\boldsymbol{G}$ are usually referred to as "internal" or "squeezing" forces. Such forces play a fundamental role in controlling manipulation tasks when Coulomb-type limitations on frictional forces are in order.

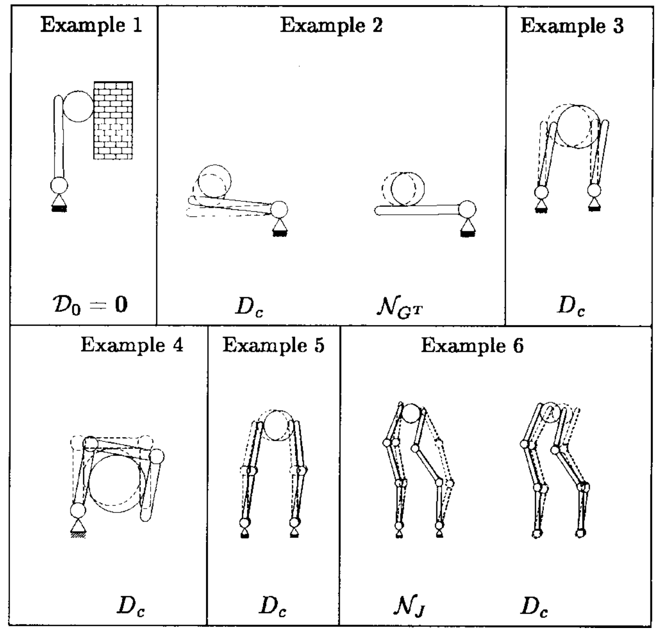

Fig. 4. Representative motions for the subsets defined in Section II.

Definition 4: A grasp is said kinematically *defective* if $\ker \boldsymbol{J}^{\mathrm{T}} \neq \mathbf{0}$.

In a defective system, there exist constraint reactions which do not influence the manipulator dynamics, cf. (2). Since $\boldsymbol{J}^{\mathrm{T}} \in \mathbb{R}^{q\times t}$, whenever the manipulation system has less degrees of freedom (DoF's) q than the number t of contact constraints, it exhibits a defective grasp.

A. Hyperstatic Grasps

The rigid-body dynamics equation (2) can be rewritten as

$$\boldsymbol{M}_{\mathrm{dyn}}\begin{bmatrix}\ddot{\boldsymbol{q}}\\ \ddot{\boldsymbol{u}}\\ \mathrm{t}\end{bmatrix} = \begin{bmatrix}\tau - \boldsymbol{Q_h}\\ \boldsymbol{w} - \boldsymbol{Q_o}\\ \boldsymbol{Q_c}\end{bmatrix} \tag{3}$$

where

$$\boldsymbol{M}_{\mathrm{dyn}} = \begin{bmatrix}\boldsymbol{M_h} & 0 & \boldsymbol{J}^{\mathrm{T}}\\ 0 & \boldsymbol{M_o} & -\boldsymbol{G}\\ \boldsymbol{J} & -\boldsymbol{G}^{\mathrm{T}} & 0\end{bmatrix}. \tag{4}$$

In order for this equation to completely determine the law of motion of the system, it is necessary that matrix $\boldsymbol{M}_{\mathrm{dyn}}$ be invertible. Such case is considered in detail by Murray, Li, and Sastry [23], who discussed the dynamics of multifinger manipulation in the hypotheses that the hand Jacobian is full row rank. For all manipulation systems with noninvertible $\boldsymbol{M}_{\mathrm{dyn}}$, the rigid-body dynamics (3) fails to determine the law of motion of the whole system. By observing that

$$\ker \boldsymbol{M}_{\mathrm{dyn}} = \{(\ddot{\boldsymbol{q}},\ddot{\boldsymbol{u}},\mathrm{t})^{\mathrm{T}} | \ddot{\boldsymbol{q}} = \mathbf{0}, \ddot{\boldsymbol{u}} = \mathbf{0}, \mathrm{t} \in \ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G}\}$$

the following definition naturally ensues.

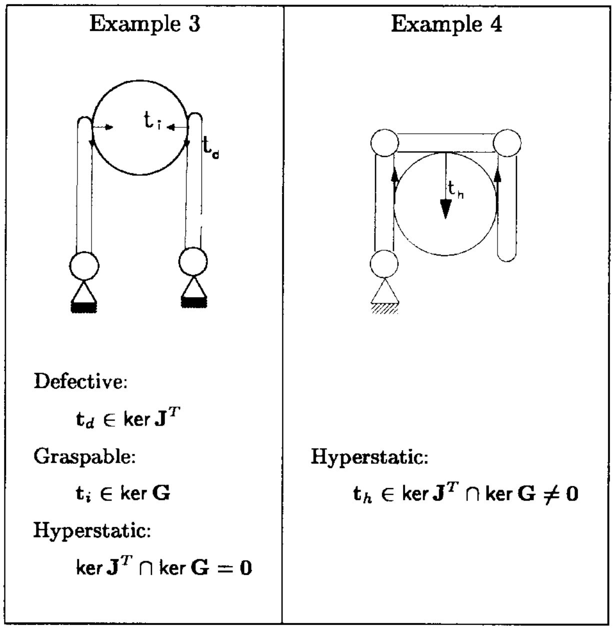

Fig. 5. Examples of defective, graspable and hyperstatic grasps. Numerical analysis has been reported in Appendix C.

Definition 5: A grasp is said *hyperstatic* if

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} \neq \mathbf{0}.$$

Defectivity and graspability are necessary conditions for the hyperstaticity of a manipulation system. Note that the above definition of hyperstaticity could be obtained also by quasistatic arguments, as for instance in [37], who found that $\mathrm{t} > q + d$ is a sufficient condition for hyperstaticity.

Fig. 5 pictorially describes the notions of defectivity, graspability and hyperstaticity for Examples 3 and 4 of Fig. 1.

Rigid-body dynamics are not satisfactory to the purposes of this paper. In fact, many interesting manipulation systems are indeed hyperstatic, e.g., whole-arm robots, and the rigid-body modelization would leave the system dynamics undetermined. Moreover rigid-body dynamics do not allow proper modelization, and hence control, of contact forces (closed-loop control of forces would in fact entail algebraic loops). Because contact force control is a central point in grasping, this is certainly an important drawback of the rigid body dynamics approach. Finally, systems with significant inherent compliance are sometimes encountered, especially in applications where stable and accurate force control is of concern.

To address such more general cases, we introduce a lumped-parameter compliant model for the hand-object dynamics (see Appendix A). In such model, Lagrange multipliers t are interpreted as constraint forces deriving from generalized virtual springs $\boldsymbol{K_i}$ and dampers $\boldsymbol{B_i}$, whose endpoints are though of as attached at the ith contact points on the object and on the finger, respectively, and are loaded by compenetration $\delta \boldsymbol{c}$ of the two bodies. The model of a general manipulation systems we will refer to is therefore

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}_h^{-1}(-\boldsymbol{Q}_h - \boldsymbol{J}^T \mathrm{t} + \tau); \tag{5}$$

$$\ddot{\boldsymbol{u}} = \boldsymbol{M}_o^{-1}(-\boldsymbol{Q}_o + \boldsymbol{G}\mathrm{t} + \boldsymbol{w}) \tag{6}$$

$$\mathrm{t} = \boldsymbol{K}\delta \boldsymbol{c} + B\delta \dot{\boldsymbol{c}}. \tag{7}$$

For the analysis of most of the structural properties of general manipulation systems, the model (5)–(7) is still intractable. Henceforth, then, we will deal with the linearized dynamic model

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_\tau \tau' + \boldsymbol{B_w}\boldsymbol{w}' \tag{8}$$

where the state vector $\boldsymbol{x} \in \mathbb{R}^{2(\boldsymbol{q}+\boldsymbol{d})}$, inputs $\tau' \in \mathbb{R}^q$, and disturbances $\boldsymbol{w}' \in \mathbb{R}^{\boldsymbol{d}}$ are defined as the departures from a reference equilibrium configuration

$$\boldsymbol{x}_{\mathrm{eq}} = [\boldsymbol{q}_{\mathrm{eq}}^{\mathrm{T}} \quad \boldsymbol{u}_{\mathrm{eq}}^{\mathrm{T}} \quad \mathbf{0}^{\mathrm{T}} \quad \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$$

at which contact forces are $\mathrm{t}(\boldsymbol{x}_{\mathrm{eq}}) = \mathrm{t}_{\mathrm{eq}}$, as

$$\begin{aligned} \boldsymbol{x} &= [\delta \boldsymbol{q}^{\mathrm{T}} \quad \delta \boldsymbol{u}^{\mathrm{T}} \quad \dot{\boldsymbol{q}}^{\mathrm{T}} \quad \dot{\boldsymbol{u}}^{\mathrm{T}}]^{\mathrm{T}} \\ &= [(\boldsymbol{q} - \boldsymbol{q}_{\mathrm{eq}})^{\mathrm{T}} \quad (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{eq}})^{\mathrm{T}} \quad \dot{\boldsymbol{q}}^{\mathrm{T}} \quad \dot{\boldsymbol{u}}^{\mathrm{T}}]^{T} \\ \tau' &= \tau - \boldsymbol{J}^{\mathrm{T}}\mathrm{t}_{\mathrm{eq}} \\ \boldsymbol{w}' &= \boldsymbol{w} + \boldsymbol{G}\mathrm{t}_{\mathrm{eq}}. \end{aligned} \tag{9}$$

Under the assumptions reported in Appendix A, the dynamics matrix $\boldsymbol{A}$, joint torque input matrix $\boldsymbol{B}_\tau$, and external wrench disturbance matrix $\boldsymbol{B_w}$ have the form (see Appendix A)

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{L_k} & -\boldsymbol{L_b} \end{bmatrix}$$

$$\boldsymbol{B}_\tau = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{M}_{\boldsymbol{h}}^{-1} \\ \mathbf{0} \end{bmatrix}; \quad \boldsymbol{B_w} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{M}_{\boldsymbol{o}}^{-1} \end{bmatrix} \tag{10}$$

where

$$\boldsymbol{L_k} = \boldsymbol{M}^{-1}\boldsymbol{P_k}; \quad \boldsymbol{L_b} = \boldsymbol{M}^{-1}\boldsymbol{P_b} \tag{11}$$

and

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} \boldsymbol{M_h} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M_o} \end{bmatrix} \\ \boldsymbol{P_k} &= \begin{bmatrix} \boldsymbol{J}^{\mathrm{T}} \\ -\boldsymbol{G} \end{bmatrix} K[\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}] \\ \boldsymbol{P_b} &= \begin{bmatrix} \boldsymbol{J}^{\mathrm{T}} \\ -\boldsymbol{G} \end{bmatrix} B[\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}]. \end{aligned} \tag{12}$$

III. Stability

The importance of the study of stability to the theory and the practice of robotic grasping is witnessed by the relatively large attention devoted to the topic in the robotics literature.

The characteristic polynomial of the linearized system is

$$\begin{aligned} \det(s\boldsymbol{I} - \boldsymbol{A}) &= \det \begin{bmatrix} s\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{L_k} & s\boldsymbol{I} + \boldsymbol{L_b} \end{bmatrix} \\ &= s^{q+d} \det(s^{-1}\boldsymbol{I}(s^2\boldsymbol{I} + s\boldsymbol{L_b} + \boldsymbol{L_k})) \\ &= \det(s^2\boldsymbol{M} + s\boldsymbol{P_b} + \boldsymbol{P_k}). \end{aligned}$$

Since, from (12) $\boldsymbol{M}$ is positive definite (p.d.) and $\boldsymbol{P_k}, \boldsymbol{P_b}$ are either p.d. or positive semidefinite (p.s.d.) or the following cases are in order.

1) $\boldsymbol{P_k}$ and $\boldsymbol{P_b}$ p.d.: the eigenvalues of the linearized system lie in the open left-half-plane;

2) $\boldsymbol{P_k}$ and $\boldsymbol{P_b}$ p.s.d. (i.e., $\ker[\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}] \neq \boldsymbol{0}$): the eigenvalues of the linearized system lie in the in the union of the open left-half-plane and the origin.

From (10)–(12), the eigenspace associated with the possible eigenvalue in the origin corresponds to the subspace of indifferent displacements from the equilibrium configuration, defined as

$$\mathcal{D}_o = \{\boldsymbol{x} | (\delta\boldsymbol{q}^{\mathrm{T}} \delta\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}} \in \ker[\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}], \dot{\boldsymbol{q}} = \dot{\boldsymbol{u}} = \boldsymbol{0}\}. \quad (13)$$

It is interesting to consider the following decomposition of this subspace;

$$\mathcal{D}_o = \mathcal{N}_J + \mathcal{N}_{G^{\mathrm{T}}} + D_c$$

where

$$\mathcal{N}_J = \{\boldsymbol{x} | \delta\boldsymbol{q} \in \ker \boldsymbol{J}, \delta\boldsymbol{u} = \dot{\boldsymbol{q}} = \dot{\boldsymbol{u}} = \boldsymbol{0}\} \quad (14)$$

is the subspace of *redundant* joint displacements

$$\mathcal{N}_{G^{\mathrm{T}}} = \{\boldsymbol{x} | \delta\boldsymbol{q} = \boldsymbol{0}, \delta\boldsymbol{u} \in \ker \boldsymbol{G}^{\mathrm{T}}, \dot{\boldsymbol{q}} = \dot{\boldsymbol{u}} = \boldsymbol{0}\} \quad (15)$$

is the subspace of *under-actuated* object displacements. Alternatively, according to Definition 2, this subspace is referred to as *indeterminate* subspace. A general framework for studying and controlling systems exhibiting an indeterminate subspace of motions has been presented by Seto and Baillieul [33], with reference to *super-articulated* mechanisms. Furthermore

$$D_c = \mathcal{D}_o - (\mathcal{N}_J \oplus \mathcal{N}_{G^{\mathrm{T}}}) \quad (16)$$

is the subset of *coordinated* displacements.

Displacements representative of the subspaces $\mathcal{N}_J$ and $\mathcal{N}_{G^{\mathrm{T}}}$ and of the subset D_c are illustrated in Fig. 2, for the examples of Fig. 1. Note that for the manipulator of Example 1, no indifferent displacement is possible; in Example 2, there is the possibility of a combined displacement of the joint and the object (D_c) and of a displacement of the object alone $(\mathcal{N}_{G^{\mathrm{T}}})$, whose position is left quasistatically indeterminate by this device. In Examples 3–5, only coordinated displacements of joints and objects (D_c) represent possible neighboring equilibria for the system, while in Example 6, besides such combined displacements, there is the possibility of exploiting the redundancy of the fingers to displace their links without affecting the object position $(\mathcal{N}_J)$.

Due to the presence of eigenvalues with null real part (case 2), stability of the nonlinear system can not be discussed based on its linearization. However, it will be shown in the next section that redundant and coordinated displacements can be stabilized by simple independent joint controllers, thus guaranteeing the local asymptotic stability of determinate $(\ker \boldsymbol{G}^{\mathrm{T}} = \boldsymbol{0})$ manipulation systems.

As already mentioned, removing some of the assumptions above leads to a much less intelligible dynamic behavior. It can be observed for instance that the effect of rolling of the object surface on that of the links, when both are convex, is to move poles rightwards in the complex plane. Also the effect on the dynamics of nonnegligible forces at the reference equilibrium coupled with large variations of the Jacobian and/or grasp matrix about the reference configuration (i.e., of terms $\partial\boldsymbol{J}^{\mathrm{T}}\mathrm{t}_{\mathrm{eq}}/\partial\boldsymbol{q}$ and $\partial\boldsymbol{G}\mathrm{t}_{\mathrm{eq}}/\partial\boldsymbol{u}$ appearing in the dynamic matrix $\boldsymbol{A}$) may be unstabilizing. System instability may result from those effects, as is usually noticed in manipulating, e.g., a soap bar. The influence on system stability of relative curvature along with that of other parameters has been studied by Montana [22] for the special case of a grasp by two fixed fingers, and for more fingers by Howard and Kumar [14]. Unfortunately, however, in general cases the kinematic or geometric parameters of the system enter the dynamics in such a complex fashion, that only dead-reckoning of the eigenvalues of the dynamic matrix for given parameter values can be done, and little structural insight is gained by stability analysis.

IV. Controllability and Stabilizability

We recall from elementary systems theory that, for a linear system $\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}$, the subspace of states that are pointwise-controllable from inputs (denoted by $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$) corresponds to the image space of the controllability matrix, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{im}[\boldsymbol{B}, \boldsymbol{A}\boldsymbol{B}, \cdots, \boldsymbol{A}^{\boldsymbol{n}-1}\boldsymbol{B}]$. An useful geometric characterization of the controllability subspace is that it is the minimal $\boldsymbol{A}$-invariant subspace containing im $\boldsymbol{B}$.

Accordingly, the subspace of states that can be reached at a given time by using joint torques as inputs in our model can be obtained (by some rather lengthy calculation reported in [27]) as

$$\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle = \left\{ \boldsymbol{x} \;\middle|\; \begin{array}{l} \delta\boldsymbol{q}, \dot{\boldsymbol{q}} \in \mathbb{R}^{\boldsymbol{q}} \\ \\ \delta\boldsymbol{u}, \dot{\boldsymbol{u}} \in \langle \boldsymbol{M_o}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}}, \boldsymbol{M_o}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{J} \rangle \end{array} \right\}. \quad (17)$$

According to definitions of Section II and to (18), the following cases may be encountered:

Non-defective and Indeterminate: Being $\boldsymbol{J}$ full row rank (f.r.r.) and $\ker \boldsymbol{G}^{\mathrm{T}} \neq \boldsymbol{0}$, it follows:

$$\langle \boldsymbol{M_o}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}}, \boldsymbol{M}_o^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{J} \rangle = \mathrm{im}(\boldsymbol{M_o}^{-1}\boldsymbol{G}) \subset \mathbb{R}^{\boldsymbol{d}}.$$

The system is not completely controllable, the controllable subspace being

$$\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle = \{\boldsymbol{x} | \delta\boldsymbol{q}, \dot{\boldsymbol{q}} \in \mathbb{R}^{\boldsymbol{q}}, \delta\boldsymbol{u}, \dot{\boldsymbol{u}} \in \mathrm{im}(\boldsymbol{M_o}^{-1}\boldsymbol{G})\}.$$

Observe that only object displacements and velocities belonging to $\mathrm{im}(\boldsymbol{M_o}^{-1}\boldsymbol{G})$ are reachable. In particular, since $\boldsymbol{M_o}$ is p.d., the *indeterminate* subspace

$$\mathcal{X}_i = \{\boldsymbol{x} | \delta\boldsymbol{q} = \dot{\boldsymbol{q}} = \boldsymbol{0}, \delta\boldsymbol{u}, \dot{\boldsymbol{u}} \in \ker \boldsymbol{G}^{\mathrm{T}}\} \quad (18)$$

is not reachable. Notice that $\mathcal{N}_{G^T}$ in (15) is the zero-velocity section of $\mathcal{X}_i$.

Defective and Determinate: Being $\ker \boldsymbol{J}^{\mathrm{T}} \neq \boldsymbol{0}$ and $\boldsymbol{G}$ f.r.r., the system may or may not loose complete controllability, depending on the particular case considered. However, the controllability of defective systems is *generic:* the subset of kinematic, inertial, and visco-elastic parameters for which controllability is lost has zero measure in the space of parameters entering the dynamic equations. For the device in Example 1 of Fig. 1, controllability of vertical and rotational movements of the object is lost due to the particular symmetry of inertia, stiffness and damping parameters that were assumed in the introduction. The same holds for the Example 3 of Fig. 1.

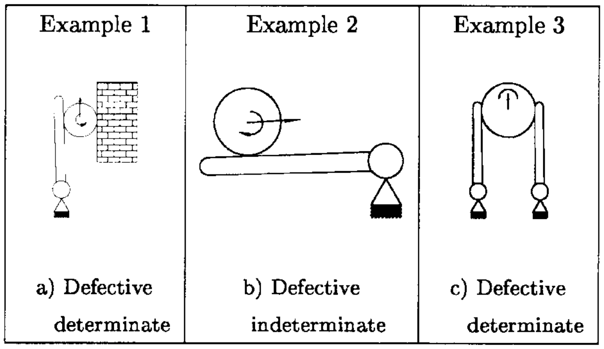

Fig. 6. Uncontrollable modes for three examples of Fig. 1.

Fig. 6 reports graphical illustrations of the uncontrollable modes in these two cases. From Appendix C it is an easy matter to verify that for the defective system of Example 4, due to the f.r.r. of $\boldsymbol{GKJ}$, the system is controllable.

Defective and Indeterminate: Being neither $\boldsymbol{J}$ nor $\boldsymbol{G}$ f.r.r., a subset of the state space is not controllable because of system indeterminacy while a different subset is generically not controllable because of the system defectivity. This is the case of Example 2 (see Fig. 6).

Non-Defective and Determinate: Being $\boldsymbol{J}$ and $\boldsymbol{G}$ f.r.r., the system is completely controllable. Such is the case for Examples 5 and 6 in Fig. 1.

Observing that the indeterminate subspace $\mathcal{X}_i$ is $\boldsymbol{A}$-invariant, and applying a state space transformation $\mathrm{T}_1 = [\mathrm{T}_\tau | \mathrm{T}_{\boldsymbol{i}} | \mathrm{T}_{\boldsymbol{d}}]$ whereof T_τ is a basis matrix[1] (b.m.) of $\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle, \mathrm{T}_i$ is a b.m. of $\mathcal{X}_i$, and $\mathrm{T}_{\boldsymbol{d}}$ is a complementary basis matrix[2] (c.b.m.) of $\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle \oplus \mathcal{X}_i$ to the state space $\mathbb{R}^{2(q+d)}$, we have that the dynamics of general robotic systems can be rewritten in the (controllability) form

$$\mathrm{T}_1^{-1} \boldsymbol{A} \mathrm{T}_1 = \left[\begin{array}{c|c|c} {}^\tau\boldsymbol{A} & 0 & * \\ \hline 0 & {}^i\boldsymbol{A} & * \\ \hline 0 & 0 & {}^d\boldsymbol{A} \end{array}\right]; \quad \mathrm{T}_1^{-1} \boldsymbol{B}_\tau = \left[\begin{array}{c} \bullet \\ \hline 0 \\ \hline 0 \end{array}\right] \tag{19}$$

where the symbol "•" stands for a nonzero element, while the symbol "$*$" represents blocks that may be zero or not. Lemma 8 in Appendix B provides a convenient choice of $\mathrm{T}_{\boldsymbol{d}}$ which annihilates the $*$ elements in (19).

The above form of the dynamics of a general robotic system points out that uncontrollable modes may appear because of two reasons. The modes associated with ${}^i\boldsymbol{A}$ are the "indeterminate modes" of the system, and are strictly related to the existence of a nullspace of the transpose of the grasp matrix, in the sense that they correspond to motions left free by the grasp. Indeterminate modes are double integrators. The uncontrollable modes associated with ${}^d\boldsymbol{A}$ are the "defective modes" of the system, since a necessary condition for their existence is that the hand Jacobian has not full row rank. This case occurs in WAM systems but also in conventional robots at their kinematic singularities. Defective modes are damped periodic oscillations.

[1] $\boldsymbol{V}$ is called a basis matrix of a subspace $\mathcal{V}$ if it is f.c.r. and $\operatorname{im} \boldsymbol{V} = \mathcal{V}$.

[2] $\boldsymbol{W}$ is called a complementary basis matrix of $\mathcal{V}$ to $\mathcal{X}$ if it is f.c.r. and $\operatorname{im} \boldsymbol{W} \oplus \mathcal{V} = \mathcal{X}$.

A. Stabilizability

By introducing a constant state feedback in the form

$$\tau = \hat{\tau} - \boldsymbol{R}\boldsymbol{x}, \qquad \boldsymbol{R} = [\boldsymbol{R}_{\boldsymbol{q}} \quad \boldsymbol{R}_{\boldsymbol{u}} \quad \boldsymbol{R}_{\dot{\boldsymbol{q}}} \quad \boldsymbol{R}_{\dot{\boldsymbol{u}}}]$$

the eigenvalues of the controllable subsystem can be relocated arbitrarily while indeterminate and defective modes are unaffected by control [see (19)]. In practical applications, however, the only state variables that can be reasonably assumed to be accessible to measurement are joint displacements and rates. In fact, object position/orientations and their rates are difficult to measure and to observe. Thus, it is of practical relevance the following *restricted stabilizability* lemma.

Lemma 1: If the system is not indeterminate ($\boldsymbol{G}$ is f.r.r.), there exists a constant linear state feedback of joint displacements and rates only

$$\boldsymbol{R}' = [\boldsymbol{R}_{\boldsymbol{q}} \quad 0 \quad \boldsymbol{R}_{\dot{\boldsymbol{q}}} \quad 0]$$

such that $\boldsymbol{A}_{\boldsymbol{f}} = \boldsymbol{A} - \boldsymbol{B}_\tau \boldsymbol{R}'$ is asymptotically stable.

Proof: Remind, from Section III, that

$$\det(s\boldsymbol{I} - \boldsymbol{A}_{\boldsymbol{f}}) = \det(s^2 \boldsymbol{M} + s\boldsymbol{P}'_{\boldsymbol{b}} + \boldsymbol{P}'_{\boldsymbol{k}})$$

where

$$\boldsymbol{P}'_{\boldsymbol{k}} = \boldsymbol{P}_{\boldsymbol{k}} + \begin{bmatrix} \boldsymbol{R}_{\boldsymbol{q}} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{P}'_{\boldsymbol{b}} = \boldsymbol{P}_{\boldsymbol{b}} + \begin{bmatrix} \boldsymbol{R}_{\dot{\boldsymbol{q}}} & 0 \\ 0 & 0 \end{bmatrix}.$$

The thesis follows from $\boldsymbol{P}'_{\boldsymbol{k}}, \boldsymbol{P}'_{\boldsymbol{b}}$ being p.d. ($\boldsymbol{R}_{\boldsymbol{q}}, \boldsymbol{R}_{\dot{\boldsymbol{q}}}$ are p.d.). In fact form (12), by putting $\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}/2}\boldsymbol{K}^{1/2}$, we have

$$\begin{aligned} &\boldsymbol{x}^{\mathrm{T}} \boldsymbol{P}'_{\boldsymbol{k}} \boldsymbol{x} \\ &\quad = [\boldsymbol{x}_1^{\mathrm{T}} \quad \boldsymbol{x}_2^{\mathrm{T}}] \begin{bmatrix} \boldsymbol{J}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{J} + \boldsymbol{R}_{\boldsymbol{q}} & -\boldsymbol{J}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}} \\ -\boldsymbol{G}\boldsymbol{K}\boldsymbol{J} & \boldsymbol{G}\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \\ &\quad = (\boldsymbol{K}^{1/2}\boldsymbol{J}\boldsymbol{x}_1 - \boldsymbol{K}^{1/2}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{x}_2)^{\mathrm{T}}(\boldsymbol{K}^{1/2}\boldsymbol{J}\boldsymbol{x}_1 - \boldsymbol{K}^{1/2}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{x}_2) \\ &\qquad + \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{R}_{\boldsymbol{q}}\boldsymbol{x}_1 > 0, \end{aligned}$$

and analogously for $\boldsymbol{P}'_{\boldsymbol{b}}$. ∎

B. Output Controllability

Being the goal of dextrous manipulation to control the position of the manipulated object through the contact forces with the fingers, it is natural to consider two outputs for a general manipulation system, namely the object position $\boldsymbol{u}$ and the contact force vector t. In the linearized model under consideration, from (9), (36), and (40), one has

$$\delta\boldsymbol{u} = \boldsymbol{C}_{\boldsymbol{u}}\boldsymbol{x}, \quad \boldsymbol{C}_{\boldsymbol{u}} = [0 \quad \boldsymbol{I} \quad 0 \quad 0] \tag{20}$$

$$\delta\mathrm{t} = \boldsymbol{C}_{\mathrm{t}}\boldsymbol{x}, \quad \boldsymbol{C}_{\mathrm{t}} = [\boldsymbol{K}\boldsymbol{J} \quad -\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}} \quad \boldsymbol{B}\boldsymbol{J} \quad -\boldsymbol{B}\boldsymbol{G}^{\mathrm{T}}]. \tag{21}$$

The pointwise output controllable subspace for contact forces can be evaluated (details are reported in [27]) as

$$\boldsymbol{C}_{\mathrm{t}}\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle = \langle \boldsymbol{A}, \boldsymbol{K}\boldsymbol{J} \rangle$$

where $\Lambda = -\boldsymbol{K}(\boldsymbol{J}\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}} + \boldsymbol{G}^{\mathrm{T}}\boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G})$.

As already mentioned, a particularly important concern in manipulation is to avoid slippage at the contacts by controlling internal forces. It is therefore most interesting to determine the subspace of internal forces which are actually controllable, defined as

$$\mathcal{F}_{hr} = \boldsymbol{C}_{\mathrm{t}}\langle \boldsymbol{A}, \boldsymbol{B}_{\tau}\rangle \cap \ker \boldsymbol{G}. \tag{22}$$

By doing some calculations (reported in [29]), an explicit and synthetic formula can be obtained as

$$\mathcal{F}_{hr} = \operatorname{im}((\boldsymbol{I} - \boldsymbol{K}\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{G}\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}})^{-1}\boldsymbol{G})\boldsymbol{K}\boldsymbol{J}).$$

By applying such formula to the example of Fig. 2, it is easily checked that only internal forces as in (b) are actually *controllable internal*, and the intuition that torsion of the object as in (c) can not be modified by acting on the joints is confirmed.

If object motions are considered as outputs, on the other hand, it holds

$$\begin{aligned}\boldsymbol{C}_{\boldsymbol{u}}\langle \boldsymbol{A}, \boldsymbol{B}_{\tau}\rangle &= \boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G}(\boldsymbol{C}_{\mathrm{t}}\langle \boldsymbol{A}, \boldsymbol{B}_{\tau}\rangle)\\ &= \boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G}\langle \Lambda, \boldsymbol{K}\boldsymbol{J}\rangle.\end{aligned}$$

Notice that arbitrary object positions can be reached if and only if the grasp map $\boldsymbol{G}$ is onto and the force controllability map $\boldsymbol{C}_{\mathrm{t}}\langle \boldsymbol{A}, \boldsymbol{B}_{\tau}\rangle$ is injective on $\operatorname{im}\boldsymbol{G}^{T}$.

V. Observability

The twofold definition of outputs for a manipulation system introduced above reflects in the following considerations on observability.

A. Observability from Object Motions

The subspace of states unobservable from $\boldsymbol{u}$ is evaluated by recurrently computing the rows of the observability matrix $\boldsymbol{O}_{\boldsymbol{u}}$ (see [27] for details) as

$$\ker \boldsymbol{O}_{\boldsymbol{u}} = \{\boldsymbol{x} | \delta\boldsymbol{q} \in \mathcal{V}_{\boldsymbol{h}}, \delta\boldsymbol{u} = \boldsymbol{0}, \dot{\boldsymbol{q}} \in \mathcal{V}_{\boldsymbol{h}}, \dot{\boldsymbol{u}} = \boldsymbol{0}\} \tag{23}$$

where $\mathcal{V}_h$ is the maximal $(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{J})$-invariant subspace contained in $\ker(\boldsymbol{G}\boldsymbol{K}\boldsymbol{J})$, i.e.,

$$\mathcal{V}_h = \bigcap_{i=1}^{q} \ker[\boldsymbol{G}\boldsymbol{K}\boldsymbol{J}(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{J})^{i-1}]. \tag{24}$$

According to definitions of Section II and to (23), the following remarks apply here:

Non-Graspable and Nonredundant: Since $\boldsymbol{J}$ and $\boldsymbol{G}$ are full column rank (f.c.r.), $\ker(\boldsymbol{G}\boldsymbol{K}\boldsymbol{J}) = \boldsymbol{0}$ and the system is completely observable from object motions, as in Example 2 of Fig. 1.

Non-Graspable and Redundant: $\boldsymbol{G}$ is f.c.r. and $\ker \boldsymbol{J} \neq \boldsymbol{0}$, thus the subspace unobservable from object motions is the *redundant subspace* defined as

$$\mathcal{X}_r = \{\boldsymbol{x} | \delta\boldsymbol{q} \in \ker \boldsymbol{J}, \delta\boldsymbol{u} = \boldsymbol{0}, \dot{\boldsymbol{q}} \in \ker \boldsymbol{J}, \dot{\boldsymbol{u}} = \boldsymbol{0}\}. \tag{25}$$

Notice that $\mathcal{N}_J$ in (14) is the zero-velocity section of $\mathcal{X}_r$. The existence of an unobservable subspace in redundant systems is generic.

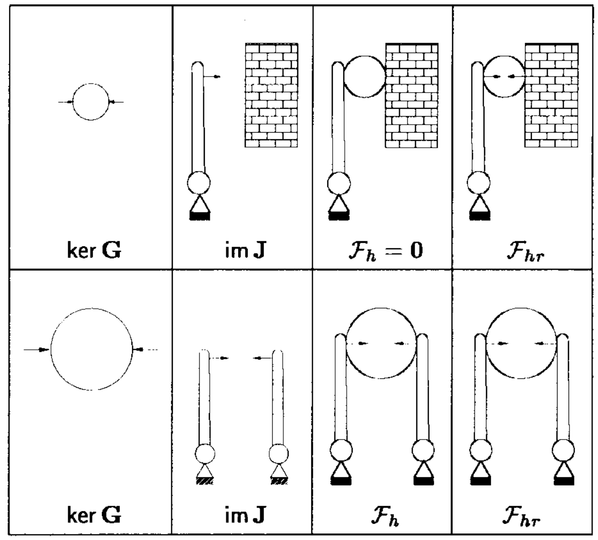

Fig. 7. Illustration of the subspace of internal contact forces ($\ker \boldsymbol{G}$) and of the subspace of dynamically internal contact forces $\mathcal{F}_h$ for the systems of Examples 1 (first row) and 3 (second row) of Fig. 1. Note that the existence of a nonzero dynamically internal subspace for Example 3 depends on the particular values assigned to geometric and inertial parameters. Reported is also illustrations of the subspace of controllable internal forces $\mathcal{F}_{hr}$, (22).

Graspable and Nonredundant: $\ker \boldsymbol{G} \neq \boldsymbol{0}$, and $\boldsymbol{J}$ is f.c.r.. The term "graspable" follows from the fact that contact forces in $\ker \boldsymbol{G}$ are usually called *grasping*, or *internal*, forces, and play a fundamental role in resisting external disturbances with unilateral friction contact constraints. The system may or may not loose complete observability from object positions, depending on the particular case considered. Observability is generic for graspable, nonredundant systems, as in Examples 1, 3–5 (see Appendix C).

Graspable and Redundant: Neither $\boldsymbol{J}$ nor $\boldsymbol{G}$ are f.c.r. (see e.g., Example 6 in Appendix C). A subspace of the state space is not observable because of redundancy, while a different one is generically not observable because of the system graspability. Notice that the subspace of redundant motions is mapped in null contact forces by $\boldsymbol{C}_{\mathrm{t}}$.

By mapping the unobservable subspace from object motions on the space of contact forces through $\boldsymbol{C}_{\mathrm{t}}$, those internal forces that can be exerted without affecting the motions of the object are obtained:

Definition 6: The subspace $\mathcal{F}_h = \boldsymbol{C}_{\mathrm{t}} \ker \boldsymbol{O}_{\boldsymbol{u}}$ is called the subspace of "dynamically internal contact forces."

The possibility of exerting internal forces without affecting the motions of the object is of great practical relevance to cases when the demand of accuracy of manipulation is highest, as for instance when the object of manipulation is a surgical tool. In the apparently similar systems of Examples 1 and 3, the possibility of exerting dynamically internal forces is illustrated. No dynamically internal force can be exerted in Example 1, being void the intersection between the column space of $\boldsymbol{J}$ and the nullspace of $\boldsymbol{G}$ as depicted in Fig. 7, first row. In Example 3, however, this intersection is not void and, due to the particular symmetry of kinematic and inertial parameters, a dynamically internal contact force can be exerted as illustrated in Fig. 7, second row.

Observing that the subspace $\mathcal{X}_r$ is $\boldsymbol{A}$-invariant, and applying a state space transformation $\mathrm{T}_2 = [\mathrm{T}_{\boldsymbol{u}} | \mathrm{T}_{\boldsymbol{r}} | \mathrm{T}_{\boldsymbol{h}}]$, whereof $\mathrm{T}_{\boldsymbol{r}}$ is a b.m. of $\mathcal{X}_r$, $\mathrm{T}_{\boldsymbol{u}}$ is a c.b.m. of $\ker \boldsymbol{O}_{\boldsymbol{u}}$ to $\mathbb{R}^{2(q+d)}$, and $\mathrm{T}_{\boldsymbol{h}}$ is a c.b.m. of $\mathcal{X}_r$ to $\ker \boldsymbol{O}_{\boldsymbol{u}}$, we have that the dynamics of general manipulation systems can be rewritten in the (observability) form

$$\mathrm{T}_2^{-1}\boldsymbol{A}\mathrm{T}_2 = \left[\begin{array}{c|c|c} {}^u\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \hline * & {}^r\boldsymbol{A} & * \\ \hline * & \boldsymbol{0} & {}^h\boldsymbol{A} \end{array}\right]$$
$$\boldsymbol{C}_{\boldsymbol{u}}\mathrm{T}_2 = [\ \bullet \ | \ \boldsymbol{0} \ | \ \boldsymbol{0} \ ]. \tag{26}$$

By Lemma 4 in Appendix B, if $\mathrm{T}_{\boldsymbol{u}}$ is chosen as a b.m. of $\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle$, and $\mathrm{T}_{\boldsymbol{h}}$ is put in the convenient form suggested by Lemma 7 in Appendix B, all the $*$ elements in (26) are annihilated.

Modes that are unobservable from object displacements may arise because of two reasons. "Redundant modes" associated with ${}^r\boldsymbol{A}$ are present whenever the Jacobian matrix has a nullspace as in Example 6. The redundant modes are double integrators, but can be arbitrarily relocated by feedback of joint variables only. The modes associated with ${}^h\boldsymbol{A}$ are called "dynamically internal modes" of the system, because of their relation with dynamically internal forces.

B. Observability from Contact Forces

The analysis of state observability from contact forces provides further insight in the kinematics of robotic systems. By recurrently computing the rows of the observability matrix from the contact forces, $\boldsymbol{O}_t$, the subspace of states unobservable from t is obtained as (see [27])

$$\ker \boldsymbol{O}_{\mathrm{t}} = \left\{ \boldsymbol{x} \left| \begin{bmatrix} \delta \boldsymbol{q} \\ \delta \boldsymbol{u} \end{bmatrix}, \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{u}} \end{bmatrix} \in \ker [\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}] \right. \right\}$$

and corresponds to displacements and velocities that leave the virtual springs and dampers unsolicited, i.e., to the rigid-body kinematics of the system.

Rigid-body kinematics are of particular interest in the control of manipulation systems. Since they do not involve visco-elastic deformations of bodies, they can be regarded as low-energy motions. In a sense, they represent the natural way to change the object posture.

Rigid kinematics can be characterized in terms of a matrix Γ whose columns form a basis for $\ker [\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}]$, and that can be written as

$$\Gamma = \begin{bmatrix} \Gamma_r & \boldsymbol{0} & \Gamma_{qc} \\ \boldsymbol{0} & \Gamma_i & \Gamma_{uc} \end{bmatrix} \tag{27}$$

where Γ_r is a b.m. of $\ker \boldsymbol{J}$, Γ_i is a b.m. of $\ker \boldsymbol{G}^{\mathrm{T}}$, and Γ_{qc} and Γ_{uc} are conformal partitions of a c.b.m. to $\ker [\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}]$ of $\mathrm{diag}\,(\ker \boldsymbol{J}, \ker \boldsymbol{G}^{\mathrm{T}})$. The analysis of the dimensions and the geometry of the subspaces spanned by the blocks of matrix Γ is instrumental in describing fundamental kinematic characteristics of robotic manipulation systems, such as the mobility, connectivity, and manipulability of manipulation systems. For instance, the structure described in Example 1 of Fig. 1 has no possible rigid motion ($\Gamma = \boldsymbol{0}$), as motions of the object may only result from deformations of the compliant elements at the contacts. For the system of Example 2, link and object motions corresponding to Γ_{qc}, Γ_{uc}, and Γ_i are pictorially represented in Fig. 8. Bicchi *et al.* [7] derived a similar description of rigid-body kinematics from quasistatic considerations, and had a detailed discussion on mobility and manipulability properties.

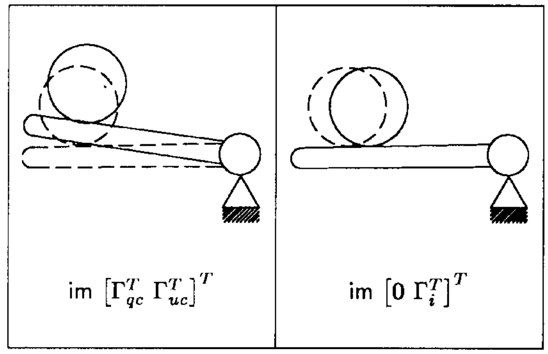

Fig. 8. Illustration of rigid body motions for Example 2 in Fig. 1.

Notice that the redundant and indeterminate subspaces, $\mathcal{X}_r$ (25) and $\mathcal{X}_i$ (18), belong to $\ker \boldsymbol{O}_{\mathrm{t}}$, thus their basis matrices $\mathrm{T}_{\boldsymbol{r}}$ and $\mathrm{T}_{\boldsymbol{i}}$ can be built in terms of Γ_r and Γ_i, respectively, as

$$\mathrm{T}_{\boldsymbol{r}} = \begin{bmatrix} \Gamma_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \Gamma_r \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \mathrm{T}_{\boldsymbol{i}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \Gamma_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \Gamma_i \end{bmatrix}. \tag{28}$$

Let $\mathrm{T}_{\boldsymbol{c}}$ be a c.b.m. of $\mathcal{X}_r \oplus \mathcal{X}_i$ to $\ker \boldsymbol{O}_{\mathrm{t}}$, in particular, according to the previous discussion, choose

$$\mathrm{T}_{\boldsymbol{c}} = \begin{bmatrix} \Gamma_{qc} & \boldsymbol{0} \\ \Gamma_{uc} & \boldsymbol{0} \\ \boldsymbol{0} & \Gamma_{qc} \\ \boldsymbol{0} & \Gamma_{uc} \end{bmatrix} \tag{29}$$

and define the subspace of *coordinated rigid* motions as $\mathcal{X}_c = \mathrm{im}\,\mathrm{T}_{\boldsymbol{c}}$. Thus it is an easy matter to verify that

$$\mathrm{im}\,[\mathrm{T}_{\boldsymbol{r}} \quad \mathrm{T}_{\boldsymbol{i}} \quad \mathrm{T}_{\boldsymbol{c}}] = \ker \boldsymbol{O}_{\mathrm{t}} \tag{30}$$

and that the column spaces of $\mathrm{T}_{\boldsymbol{c}}, \mathrm{T}_{\boldsymbol{r}}, \mathrm{T}_{\boldsymbol{i}}$ are $\boldsymbol{A}$-invariant.

Finally, applying a state space transformation $\mathrm{T}_3 = [\mathrm{T}_{\mathrm{t}} | \mathrm{T}_{\boldsymbol{r}} | \mathrm{T}_{\boldsymbol{i}} | \mathrm{T}_{\boldsymbol{c}}]$, where T_{t} is a c.b.m. of $\ker \boldsymbol{O}_{\mathrm{t}}$ to $\mathbb{R}^{2(q+d)}$, a standard observability form is obtained as

$$\mathrm{T}_3^{-1}\boldsymbol{A}\mathrm{T}_3 = \left[\begin{array}{c|c|c|c} {}^t\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \hline * & {}^r\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \hline * & \boldsymbol{0} & {}^i\boldsymbol{A} & \boldsymbol{0} \\ \hline * & \boldsymbol{0} & \boldsymbol{0} & {}^c\boldsymbol{A} \end{array}\right]$$
$$\boldsymbol{C}_{\mathrm{t}}\mathrm{T}_3 = [\ \bullet \ | \ \boldsymbol{0} \ | \ \boldsymbol{0} \ | \ \boldsymbol{0} \ ].$$

Modes that are unobservable from contact forces arise whenever the $\ker [\boldsymbol{J} \quad -\boldsymbol{G}^{\mathrm{T}}]$ is nonzero, i.e., whenever there exist rigid-body redundant, indeterminate or coordinate motions.

VI. Standard Forms

The dynamic structure of a general manipulation system, analyzed from different viewpoints in the preceding sections, is summarized in this section in a standard form of the dynamics equations. As a necessary preliminary, however, we briefly consider here the dual properties to controllability from joint torques and to observability from object displacements that were discussed previously. Such duals are observability from the position of joints $\boldsymbol{q}$ (i.e., with output matrix $\boldsymbol{C_q} = [\boldsymbol{I} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0}]$), and controllability from disturbances $\boldsymbol{w}$ (i.e., with input matrix $\boldsymbol{RB_w} = [0 \quad 0 \quad 0 \quad \boldsymbol{M_o}^{-1}]$), respectively. In fact, one has

$$\ker \boldsymbol{O_q} = \{\boldsymbol{x} | \delta \boldsymbol{q} = \boldsymbol{0}, \delta \boldsymbol{u} \in \mathcal{W}_{\boldsymbol{h}}, \dot{\boldsymbol{q}} = \boldsymbol{0}, \dot{\boldsymbol{u}} \in \mathcal{W}_{\boldsymbol{h}}\} \tag{31}$$

where

$$\mathcal{W}_h = \bigcap_{i=1}^{d} \ker [\boldsymbol{J}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{G}^{\mathrm{T}} (\boldsymbol{M_o}^{-1} \boldsymbol{G} \boldsymbol{K} \boldsymbol{G}^{\mathrm{T}})^{i-1}] \tag{32}$$

and

$$\langle \boldsymbol{A}, \boldsymbol{B_w} \rangle = \left\{ \boldsymbol{x} \;\middle|\; \begin{array}{l} \delta \boldsymbol{q}, \dot{\boldsymbol{q}} \in \langle \boldsymbol{M_h}^{-1} \boldsymbol{J}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{J}, \boldsymbol{M_h}^{-1} \boldsymbol{J}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{G}^{\mathrm{T}} \rangle \\ \delta \boldsymbol{u}, \dot{\boldsymbol{u}} \in \mathbb{R}^{\boldsymbol{d}} \end{array} \right\}. \tag{33}$$

All structural properties of general manipulation systems concerning controllability and observability are summarized in the following theorem which provides a rather interesting standard form of the dynamics of general manipulation systems. Consider a new basis of the state space defined as

$$\mathrm{T} = [\mathrm{T}_{\boldsymbol{r}} \quad \mathrm{T}_{\boldsymbol{h}} \quad \mathrm{T}_{\mathrm{o}} \quad \mathrm{T}_{\boldsymbol{i}} \quad \mathrm{T}_{\boldsymbol{d}}]$$

where T_i is defined as in Section IV, $\mathrm{T}_{\boldsymbol{r}}$ as in Section V-A, $\mathrm{T}_{\boldsymbol{h}}$ and $\mathrm{T}_{\boldsymbol{d}}$ as in Appendix B, and $\mathrm{T}_{\boldsymbol{o}}$ is a b.m. of the $\boldsymbol{A}$-invariant subspace $\langle \boldsymbol{A}, \boldsymbol{B}_\tau \rangle \cap \langle \boldsymbol{A}, \boldsymbol{B_w} \rangle$.

Theorem 1: In the coordinates $\boldsymbol{z} = \mathrm{T}^{-1} \boldsymbol{x}$, system matrices of the linearized dynamics takes on the standard form seen on the bottom of the page.

The subspace im $\mathrm{T}_{\boldsymbol{o}}$ is controllable from joint torques and external wrenches, and observable from object positions and outputs (complete observability from contact forces is not guaranteed). We also recall that $\mathrm{T}_{\boldsymbol{i}}$ and $\mathrm{T}_{\boldsymbol{r}}$ are bases of the subspaces of indeterminate and redundant motions, while $\mathrm{T}_{\boldsymbol{h}}$ and $\mathrm{T}_{\boldsymbol{d}}$ are bases of the subspaces that generate dynamically internal forces and noncontrollable (defective) forces, respectively.

The description of the geometric structure of the state space of general manipulation systems offered by Theorem 1 can be further refined by studying the dynamics of coordinated rigid motions. As already pointed out, coordinated rigid motions are of fundamental importance in manipulation and therefore their analysis deserves particular attention. In the most general case, due to possible gyroscopic effects, coordinated rigid modes may not be dynamically decoupled from redundant and/or indeterminate modes, and may be not completely controllable and/or observable. However, Lemmas 9 and 10 in Appendix B offer necessary and sufficient conditions for such dynamic decoupling indeed to occur.

A finer decomposition of the state space can be achieved in this case as

$$\hat{\mathrm{T}} = [\mathrm{T}_{\boldsymbol{r}} \quad \mathrm{T}_{\boldsymbol{h}} \quad \mathrm{T}_{\boldsymbol{c}} \quad \mathrm{T}_{\boldsymbol{a}} \quad \mathrm{T}_{\boldsymbol{i}} \quad \mathrm{T}_{\boldsymbol{d}}]$$

where $\mathrm{T}_{\boldsymbol{c}}$ is defined as in Section V-B, and $\mathrm{T}_{\boldsymbol{a}}$ is a c.b.m. of im $\mathrm{T}_{\boldsymbol{c}}$ to im $\mathrm{T}_{\boldsymbol{o}}$, such that im $\mathrm{T}_{\boldsymbol{a}} = \boldsymbol{M}^{-1} \operatorname{im} \mathrm{T}_{\boldsymbol{c}}^{\perp} \cap \operatorname{im} \mathrm{T}_{\boldsymbol{o}}$. To such decomposition corresponds a second, more articulated standard form:

Theorem 2: The condition

$$\operatorname{im} \begin{bmatrix} \Gamma_{qc} \\ \Gamma_{uc} \end{bmatrix} \subseteq \boldsymbol{M}^{-1} \operatorname{im} \begin{bmatrix} \boldsymbol{J}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G} \end{bmatrix} \tag{34}$$

is necessary and sufficient for the linearized dynamics in the new coordinates $\boldsymbol{z} = \hat{\mathrm{T}}^{-1} \boldsymbol{x}$, to take on the form shown on the bottom of the next page.

We remark here that im $\mathrm{T}_{\boldsymbol{c}}$ is the subspace of all rigid-body coordinated motions of the system, while im $\mathrm{T}_{\boldsymbol{a}}$ is controllable from any of the inputs and observable from any of the outputs considered, and corresponds to motions in presence of elastic deformations.

Structural properties of the dynamics, highlighted by Theorems 1 and 2, for the examples of Fig. 1 are reported in Appendix C.

$$\mathrm{T}^{-1}\boldsymbol{A}\mathrm{T} = \left[\begin{array}{c|c|c|c|c} {}^{r}\boldsymbol{A} & 0 & 0 & 0 & 0 \\ \hline 0 & {}^{h}\boldsymbol{A} & 0 & 0 & 0 \\ \hline 0 & 0 & {}^{o}\boldsymbol{A} & 0 & 0 \\ \hline 0 & 0 & 0 & {}^{i}\boldsymbol{A} & 0 \\ \hline 0 & 0 & 0 & 0 & {}^{d}\boldsymbol{A} \end{array}\right] \mathrm{T}^{-1}\boldsymbol{B}_\tau = \left[\begin{array}{c} \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline 0 \\ \hline 0 \end{array}\right] \quad \mathrm{T}^{-1}\boldsymbol{B_w} = \left[\begin{array}{c} 0 \\ \hline 0 \\ \bullet \\ \hline \bullet \\ \hline \bullet \end{array}\right]$$

$$\begin{aligned} \boldsymbol{C_u}\mathrm{T} &= [\ 0 \ | \ 0 \ | \ \bullet \ | \ \bullet \ | \ \bullet \ ] \\ \boldsymbol{C_q}\mathrm{T} &= [\ \bullet \ | \ \bullet \ | \ \bullet \ | \ 0 \ | \ 0 \ ] \\ \boldsymbol{C}_{\mathrm{t}}\mathrm{T} &= [\ 0 \ | \ \bullet \ | \ \odot \ | \ 0 \ | \ \bullet \ ]. \end{aligned}$$

VII. Discussion

The standard forms in which the dynamics of a general manipulation system can be written by using suitable coordinates for representing its states, as reported in Theorems 1 and 2, summarizes most results of this paper, and represents perhaps its main contribution. Such form synthetically contains information relating to the structural properties of the various subsystems. It can be seen, for instance, that the free evolution of the system from nonzero initial conditions belonging to any one of the fundamental subspaces (redundant, dynamically internal, indeterminate, and defective), remains inside the same subspace. In other words, the fundamental modes are dynamically decoupled and can be independently excited. In the standard forms, presence of a $\bullet$ block in an input (output) matrix indicates the controllability (observability) of the corresponding subsystem, while a zero block indicates the lack of the same property (this comes easily by applying the P.B.H. test).

In most part of this paper (except for Section IV-A), the manipulation system has been assumed in open-loop, or the feedback gains were assumed to be fixed and given. However, exploitation of feedback design (in particular, of the available states through matrices $\boldsymbol{R_q}$ and $\boldsymbol{R_{\dot{q}}}$) in order to modify the geometry of the system and match particular task specifications, such as e.g., disturbance decoupling or noninteracting control, is a most interesting and promising extension of the approach followed in this paper [29], [28].

Being robotic systems highly nonlinear in nature, one may question the validity of the linearization approach to the analysis. The simplicity of results achievable by linearization appears to be important at this rather early stage of investigation of complex manipulation systems. Moreover, it is well known that some of the results on the linearized system (e.g., asymptotic stability and pointwise controllability) imply analogous local properties for the real system. Conditions on the linearized system are only sufficient in general, and wider applicability of some property may hold for the nonlinear system. This is the case for instance when constraints of nonholonomic type are present (as it happens when considering rolling in 3-D between fingers and objects). In fact, driftless nonholonomic systems may exhibit complete controllability over a state-space with higher dimension than the number of inputs, which fact is clearly not possible for their linearized counterpart.

While further efforts are necessary to capture the wealth of possibilities offered by the nonlinear nature of the problem, the tools developed in this paper on the linearized model are believed to be useful in the design of control algorithms for general manipulation systems notwithstanding the intrinsic nonlinearity of robotic devices.

Appendix A

We discuss in some detail the contact constraints (1), the lumped-parameter visco-elastic model (5)–(6) of dynamics and its linearization (8)–(11) about equilibrium configurations.

A. Rigid-Body Contact Constraints

A hand-object system is a constrained mechanical system, whose dynamical description can be derived using Euler-Lagrange's equations along with constraint equations.

The disjoint dynamics of the hand and of the object are written as

$$\left(\frac{d}{dt}\frac{\partial L_h(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L_h(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}\right)^{\mathrm{T}} = \boldsymbol{M}_h(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{Q_h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \tau$$

$$\left(\frac{d}{dt}\frac{\partial L_o(\boldsymbol{u},\dot{\boldsymbol{u}})}{\partial \dot{\boldsymbol{u}}} - \frac{\partial L_o(\boldsymbol{u},\dot{\boldsymbol{u}})}{\partial \boldsymbol{u}}\right)^{T} = \boldsymbol{M}_o(\boldsymbol{u})\ddot{\boldsymbol{u}} + \boldsymbol{Q}_o(\boldsymbol{u},\dot{\boldsymbol{u}}) = \boldsymbol{w}$$

where $L_h(\cdot,\cdot)$ and $L_o(\cdot,\cdot)$ are the Lagrangians, $\boldsymbol{M}_h(\cdot)$ and $\boldsymbol{M}_o(\cdot)$ are symmetric and p.d. inertia matrices and $\boldsymbol{Q}_h(\cdot,\cdot)$ and $\boldsymbol{Q}_o(\cdot,\cdot)$ are terms including velocity-dependent and gravity forces of the hand and of the object, respectively.

Hand and object dynamics are linked through n rigid-body contact constraints, i.e., unilateral constraints of the type

$$\boldsymbol{C_i}(\boldsymbol{q},\boldsymbol{u},\dot{\boldsymbol{q}},\dot{\boldsymbol{u}}) \geq \boldsymbol{0}, \qquad \boldsymbol{i} = \boldsymbol{1},\cdots,\boldsymbol{n}.$$

$$\hat{\mathrm{T}}^{-1}\boldsymbol{A}\hat{\mathrm{T}} = \left[\begin{array}{c|c|c|c|c|c} {}^r\boldsymbol{A} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & {}^h\boldsymbol{A} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & {}^c\boldsymbol{A} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & {}^a\boldsymbol{A} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & {}^i\boldsymbol{A} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & {}^d\boldsymbol{A} \end{array}\right] \quad \hat{\mathrm{T}}^{-1}\boldsymbol{B}_\tau = \left[\begin{array}{c} \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline 0 \\ \hline 0 \end{array}\right] \quad \hat{\mathrm{T}}^{-1}\boldsymbol{B_w} = \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \end{array}\right]$$

$$\boldsymbol{C_u}\hat{\mathrm{T}} = [\ 0 \mid 0 \mid \bullet \mid \bullet \mid \bullet \mid \bullet\ ]$$

$$\boldsymbol{C_q}\hat{\mathrm{T}} = [\ \bullet \mid \bullet \mid \bullet \mid \bullet \mid 0 \mid 0\ ]$$

$$\boldsymbol{C}_{\mathrm{t}}\hat{\mathrm{T}} = [\ 0 \mid \bullet \mid 0 \mid \bullet \mid 0 \mid \bullet\ ].$$

This inequality relationship reflects the fact that the ith contact can be lost if the contacting bodies are brought away from each other. This involves an abrupt change of the structure of the model under consideration. To avoid inessential difficulties, we assume that manipulation is studied during time intervals when constraints hold with the equality sign. Assuming all contact constraints are either holonomic or Pfaffian, the whole set of contact constraints can be rewritten as

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{u})\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{u}}\end{bmatrix}=\boldsymbol{0}.$$

Several types of contact models can be used to describe the interaction between the links and the object, among which the most useful are probably the point-contact-with-friction model (or "hard-finger"), the "soft-finger" model, and the complete-constraint model (or "very-soft-finger") [31], [11]. In each case, the ith contact constraint consists in imposing that some components of the relative velocity between the surfaces are zero

$$\boldsymbol{H_i}({}^h\dot{\boldsymbol{c}}_i-{}^o\dot{\boldsymbol{c}}_i)=\boldsymbol{0} \tag{35}$$

where ${}^h\boldsymbol{c_i}, {}^o\boldsymbol{c_i}$ are d-dimensional vectors locally describing the position and orientation of a frame attached to the surface of the robot link and of the object, respectively. These frames are centered at the contact point and oriented according to the Gauss frame rule, being regularity of surfaces taken for granted. The selection matrix $\boldsymbol{H}_i$ is constant and depends on the model assumed for the ith contact. Details on the construction of matrix $\boldsymbol{H}$ can be found, e.g., in [7].

As the two Gauss reference frames are fixed on the object and the robot, respectively, their velocities can be expressed as a linear function of the velocities of the object and of the manipulator joints as

$${}^o\dot{\boldsymbol{c}}_i=\tilde{\boldsymbol{G}}_i^{\mathrm{T}}\dot{\boldsymbol{u}};\quad {}^h\dot{\boldsymbol{c}}_i=\tilde{\boldsymbol{J}}_i\dot{\boldsymbol{q}}. \tag{36}$$

Notice that the elements of $\tilde{\boldsymbol{G}}_i$ depend in general upon $\boldsymbol{u}$ and on (a parametrization of) the position of the ith point of contact on the object surface; and analogously for $\tilde{\boldsymbol{J}}_i$ upon $\boldsymbol{q_i}$ and on the position of the ith contact on the robot link.

Similar relationships hold for each contact point, and a single equation can be built to represent all (say t) the contact constraints of the system. From (35) and (36), by properly juxtaposing vectors and block matrices, the constraint matrix $\boldsymbol{C}(\boldsymbol{q},\boldsymbol{u})$ takes on the following form:

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{u})\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{u}}\end{bmatrix}=[\boldsymbol{J}\quad -\boldsymbol{G}^{\mathrm{T}}]\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{u}}\end{bmatrix}=\boldsymbol{0}$$

where matrices $\boldsymbol{G}=\tilde{\boldsymbol{G}}\boldsymbol{H}^{\mathrm{T}}\in\mathbb{R}^{\boldsymbol{d}\times\mathrm{t}}$ and $\boldsymbol{J}=\boldsymbol{H}\tilde{\boldsymbol{J}}\in\mathbb{R}^{\mathrm{t}\times\boldsymbol{q}}$ are customarily referred to as "grasp matrix" and "hand Jacobian," respectively.

B. A Lumped-Parameter Compliant Model

To address hyperstatic manipulation systems, it is necessary to introduce further structure in the mechanical model, namely, elastic energy terms

$$K_i=\tfrac{1}{2}\,\xi_i^{\mathrm{T}}({}^o\boldsymbol{c_i},{}^h\boldsymbol{c_i})\boldsymbol{K_i}\xi_i({}^o\boldsymbol{c_i},{}^h\boldsymbol{c_i})$$

and dissipation terms

$$B_i=\tfrac{1}{2}\,\dot{\xi}_i^{\mathrm{T}}({}^o\boldsymbol{c_i},{}^h\boldsymbol{c_i},{}^o\dot{\boldsymbol{c}}_{\boldsymbol{i}},{}^h\dot{\boldsymbol{c}}_{\boldsymbol{i}})\boldsymbol{B_i}\dot{\xi}_{\boldsymbol{i}}({}^o\boldsymbol{c_i},{}^h\boldsymbol{c_i},{}^o\dot{\boldsymbol{c}}_{\boldsymbol{i}},{}^h\dot{\boldsymbol{c}}_{\boldsymbol{i}})$$

where $\boldsymbol{K_i},\boldsymbol{B_i}$ are symmetric, positive definite matrices incorporating (hand/object) material "stiffness" and "damping" characteristics, and $\xi_i(\cdot,\cdot)$ is a suitable displacement function[3] applied to the position of the Gauss frames on the object and finger surfaces at the ith contact point.

Having included the elastic energy and dissipation terms in the model of the whole system, the standard derivation of the now decoupled dynamics can now be applied and gives

$$\boldsymbol{M}_h\ddot{\boldsymbol{q}}+\boldsymbol{Q}_h+\left[\frac{\partial\xi}{\partial^o\boldsymbol{c}}\frac{\partial^o\boldsymbol{c}}{\partial\boldsymbol{q}}+\frac{\partial\xi}{\partial^h\boldsymbol{c}}\frac{\partial^h\boldsymbol{c}}{\partial\boldsymbol{q}}\right]^{\mathrm{T}}\boldsymbol{K}\xi$$

$$+\left[\frac{\partial\dot{\xi}}{\partial^o\dot{\boldsymbol{c}}}\frac{\partial^o\dot{\boldsymbol{c}}}{\partial\dot{\boldsymbol{q}}}+\frac{\partial\dot{\xi}}{\partial^h\dot{\boldsymbol{c}}}\frac{\partial^h\dot{\boldsymbol{c}}}{\partial\dot{\boldsymbol{q}}}\right]^{\mathrm{T}}\boldsymbol{B}\dot{\xi}=\tau \tag{37}$$

$$\boldsymbol{M}_o\ddot{\boldsymbol{u}}+\boldsymbol{Q}_o+\left[\frac{\partial\xi}{\partial^o\boldsymbol{c}}\frac{\partial^o\boldsymbol{c}}{\partial\boldsymbol{u}}+\frac{\partial\xi}{\partial^h\boldsymbol{c}}\frac{\partial^h\boldsymbol{c}}{\partial\boldsymbol{u}}\right]^{\mathrm{T}}\boldsymbol{K}\xi \tag{38}$$

$$+\left[\frac{\partial\dot{\xi}}{\partial^o\dot{\boldsymbol{c}}}\frac{\partial^o\dot{\boldsymbol{c}}}{\partial\dot{\boldsymbol{u}}}+\frac{\partial\dot{\xi}}{\partial^h\dot{\boldsymbol{c}}}\frac{\partial^h\dot{\boldsymbol{c}}}{\partial\dot{\boldsymbol{u}}}\right]^{\mathrm{T}}\boldsymbol{B}\dot{\xi}=\boldsymbol{w} \tag{39}$$

where $\boldsymbol{K}$ and $\boldsymbol{B}$ are the aggregated stiffness and damping matrices for the manipulation system. Computation of these matrices based on knowledge of visco-elastic parameters of contacting bodies is possible along the lines of [12]. Although in practice such knowledge might be difficult to obtain, procedures similar to those currently used to identify inertial parameters of robot arms can be conceivably used to estimate visco-elastic parameters.

The following assumptions are introduced.

A1: $\xi({}^h\boldsymbol{c},{}^o\boldsymbol{c})=\boldsymbol{H}({}^h\boldsymbol{c}-{}^o\boldsymbol{c})$. This amounts to assuming a linear elastic model for the bodies.

A2: Contact points do not change by rolling. From (36) and from the identity

$$\frac{\partial^o\dot{\boldsymbol{c}}}{\partial\dot{\boldsymbol{u}}}=\frac{\partial}{\partial\dot{\boldsymbol{u}}}\left(\frac{\partial^o\boldsymbol{c}}{\partial\mathrm{t}}+\frac{\partial^o\boldsymbol{c}}{\partial\boldsymbol{u}}\dot{\boldsymbol{u}}\right)=\frac{\partial^o\boldsymbol{c}}{\partial\boldsymbol{u}}$$

one gets $(\partial^o\boldsymbol{c}/\partial\boldsymbol{u})=\tilde{\boldsymbol{G}}^{\mathrm{T}}(\boldsymbol{u})$. Similarly, $(\partial^h\boldsymbol{c}/\partial\boldsymbol{q})=\tilde{\boldsymbol{J}}(\boldsymbol{q})$. Further, $(\partial^h\boldsymbol{c}/\partial\boldsymbol{u})=(\partial^o\boldsymbol{c}/\partial\boldsymbol{q})=\boldsymbol{0}$. Non-rolling contacts can be reasonably assumed when the relative curvature of the contacting bodies is high. Neglecting the effects of rolling pairs affects the generality of the following results, as will be remarked later. Rather than by the mathematical difficulties in dealing with rolling contacts (that can be satisfactorily treated in a rigid-body setting, see, e.g., [4], [23], this assumption is motivated by the lack of a tractable model of rolling and compliant contacts [16].

In this setting, the Lagrange multipliers t can be interpreted as representing the vector of constraint forces deriving from virtual "springs" and "dampers" with endpoints attached at the contact points ${}^o\boldsymbol{c_i}$'s and ${}^h\boldsymbol{c_i}$'s. Denoting the displacement at

[3]The proper choice of this displacement function is actually a hard problem in the analysis of contact mechanics, see e.g., [16]. A detailed discussion of this point may be found in [34].

the ith contact by $\delta \boldsymbol{c_i} = \boldsymbol{H_i}({}^h\boldsymbol{c_i} - {}^o\boldsymbol{c_i})$, we have therefore

$$\begin{aligned} \mathrm{t}_{\boldsymbol{i}} &= \boldsymbol{K_i H_i}({}^h\boldsymbol{c_i} - {}^o\boldsymbol{c_i}) + \boldsymbol{B_i H_i}({}^h\dot{\boldsymbol{c}}_{\boldsymbol{i}} - {}^o\dot{\boldsymbol{c}}_{\boldsymbol{i}}) \\ &= \boldsymbol{K_i}\delta\boldsymbol{c_i} + \boldsymbol{B_i}\delta\dot{\boldsymbol{c}}_{\boldsymbol{i}}. \end{aligned} \tag{40}$$

Accordingly, the aggregated compliant dynamic and contact force model are derived as in (5)–(7).

C. Linearization

Consider first the hand dynamics (5), and let the Coriolis and centrifugal terms matrix be denoted by $\boldsymbol{C_h}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, while the gravitational term is $\boldsymbol{v_h}(\boldsymbol{q})$, so that

$$\boldsymbol{Q}_h(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C_h}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{v_h}(\boldsymbol{q}).$$

Observe that, due to its particular structure, $\boldsymbol{C_h}(\boldsymbol{q}, \boldsymbol{0}) = \boldsymbol{0}$. The first-order approximation of the first term on the right hand side of (5) is

$$\begin{aligned} &\boldsymbol{M_h^{-1}Q_h} \\ &\approx [\boldsymbol{M_h^{-1}Q_h}]_{\mathrm{eq}} + \left[\frac{\partial \boldsymbol{M_h^{-1}Q_h}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \delta\boldsymbol{q} + \left[\frac{\partial \boldsymbol{M_h^{-1}Q_h}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} \dot{\boldsymbol{q}}. \end{aligned}$$

Since

$$\begin{aligned} [\boldsymbol{M_h^{-1}Q_h}]_{\mathrm{eq}} &= [\boldsymbol{M_h^{-1}C_h}\dot{\boldsymbol{q}} + \boldsymbol{M_h^{-1}v_h}]_{\mathrm{eq}} = [\boldsymbol{M_h^{-1}v_h}]_{\mathrm{eq}} \\ \left[\frac{\partial \boldsymbol{M_h^{-1}Q_h}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} &= \left[\frac{\partial \boldsymbol{M_h^{-1}C_h}\dot{\boldsymbol{q}}}{\partial \boldsymbol{q}} + \frac{\partial \boldsymbol{M_h^{-1}v_h}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \\ &= \left[\frac{\partial \boldsymbol{M_h^{-1}v_h}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \\ \left[\frac{\partial \boldsymbol{M_h^{-1}Q_h}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} &= \left[\boldsymbol{M_h^{-1}}\frac{\partial \boldsymbol{C_h}\dot{\boldsymbol{q}}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} = \boldsymbol{0} \end{aligned}$$

we have

$$\boldsymbol{M_h^{-1}Q_h} \approx [\boldsymbol{M_h^{-1}v_h}]_{\mathrm{eq}} + \left[\frac{\partial \boldsymbol{M_h^{-1}v_h}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \delta\boldsymbol{q}.$$

In order to evaluate the first-order approximation of terms involving contact forces, recall that from hypotheses **A1** and **A2** it holds

$$\begin{aligned} \left[\frac{\partial \mathrm{t}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} &= [\boldsymbol{KJ}]_{\mathrm{eq}} + \left[\frac{\partial \boldsymbol{K}\xi_{\mathrm{eq}}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}}; \quad \left[\frac{\partial \mathrm{t}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} = [\boldsymbol{BJ}]_{\mathrm{eq}} \\ \left[\frac{\partial \mathrm{t}}{\partial \boldsymbol{u}}\right]_{\mathrm{eq}} &= -[\boldsymbol{KG}^{\mathrm{T}}]_{\mathrm{eq}} + \left[\frac{\partial \boldsymbol{K}\xi_{\mathrm{eq}}}{\partial \boldsymbol{u}}\right]_{\mathrm{eq}} \\ \left[\frac{\partial \mathrm{t}}{\partial \dot{\boldsymbol{u}}}\right]_{\mathrm{eq}} &= -[\boldsymbol{BG}^{\mathrm{T}}]_{\mathrm{eq}}. \end{aligned}$$

Note that in local (contact) frames, stiffness can be assumed to be invariant with the configuration. However, terms involving the derivatives of the stiffness matrix $(\boldsymbol{K})$ appear in general due to the fact that it relates contact forces and displacements in the base frame, then its representation does depend on system configuration.

The first-order approximation of the second term of the right-hand side of (5) is as follows:

$$\begin{aligned} \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t} \approx\, & [\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}]_{\mathrm{eq}} \\ &+ \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \delta\boldsymbol{q} + \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} \dot{\boldsymbol{q}} \\ &+ \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \boldsymbol{u}}\right]_{\mathrm{eq}} \delta\boldsymbol{u} + \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \dot{\boldsymbol{u}}}\right]_{\mathrm{eq}} \dot{\boldsymbol{u}} \end{aligned}$$

where

$$\begin{aligned} \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} &= \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \mathrm{t}_{\mathrm{eq}} + \left[\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\frac{\partial \mathrm{t}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \\ &= \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}}{\partial \boldsymbol{q}}\right]_{\mathrm{eq}} \mathrm{t}_{\mathrm{eq}} \\ &\quad + \left[\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\left(\boldsymbol{KJ} + \frac{\partial \boldsymbol{K}\xi_{\mathrm{eq}}}{\boldsymbol{q}}\right)\right]_{\mathrm{eq}} \\ \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} &= \left[\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\frac{\partial \mathrm{t}}{\partial \dot{\boldsymbol{q}}}\right]_{\mathrm{eq}} = [\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\boldsymbol{BJ}]_{\mathrm{eq}} \\ \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \boldsymbol{u}}\right]_{\mathrm{eq}} &= \left[\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\frac{\partial \mathrm{t}}{\partial \boldsymbol{u}}\right]_{\mathrm{eq}} \\ &= \left[-\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\left(\boldsymbol{KG}^{\mathrm{T}} - \frac{\partial \boldsymbol{K}\xi_{\mathrm{eq}}}{\boldsymbol{u}}\right)\right]_{\mathrm{eq}} \\ \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t}}{\partial \dot{\boldsymbol{u}}}\right]_{\mathrm{eq}} &= \left[\boldsymbol{M}_h^{-1}\boldsymbol{J}^{\mathrm{T}}\frac{\partial \mathrm{t}}{\partial \dot{\boldsymbol{u}}}\right]_{\mathrm{eq}} \\ &= [-\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\boldsymbol{BG}^{\mathrm{T}}]_{\mathrm{eq}}. \end{aligned}$$

Hence

$$\begin{aligned} \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\mathrm{t} \quad \approx\, & [\boldsymbol{M_h^{-1}J}^{\mathrm{T}}]_{\mathrm{eq}}\mathrm{t}_{\mathrm{eq}} \\ &+ \left[\frac{\partial \boldsymbol{M_h^{-1}J}^{\mathrm{T}}}{\partial \boldsymbol{q}}\mathrm{t} + \boldsymbol{M_h^{-1}J}^{\mathrm{T}}\left(\boldsymbol{KJ} + \frac{\partial \boldsymbol{K}\xi_{\mathrm{eq}}}{\boldsymbol{q}}\right)\right]_{\mathrm{eq}} \delta\boldsymbol{q} \\ &+ \left[-\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\left(\boldsymbol{KG}^{\mathrm{T}} - \frac{\partial \boldsymbol{K}\xi_{\mathrm{eq}}}{\boldsymbol{u}}\right)\right]_{\mathrm{eq}} \delta\boldsymbol{u} \\ &+ [-\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\boldsymbol{BG}^{\mathrm{T}}]_{\mathrm{eq}}\dot{\boldsymbol{u}} + [\boldsymbol{M_h^{-1}J}^{\mathrm{T}}\boldsymbol{BJ}]_{\mathrm{eq}}\dot{\boldsymbol{q}}. \end{aligned}$$

The last term of (5) is first-order approximated as

$$\boldsymbol{M_h^{-1}}\tau \approx [\boldsymbol{M_h^{-1}}\tau]_{\mathrm{eq}} + \left[\frac{\partial \boldsymbol{M_h^{-1}}}{\partial \boldsymbol{q}}\tau\right]_{\mathrm{eq}} \delta\boldsymbol{q} + [\boldsymbol{M_h^{-1}}]_{\mathrm{eq}}\delta\tau.$$

Recalling that, at the equilibrium

$$\boldsymbol{v_h} + \boldsymbol{J}^{\mathrm{T}}\mathrm{t} - \tau = \boldsymbol{0}$$

and proceeding to some simplification, we finally get the first-order approximation of the hand dynamics (5)

$$\begin{aligned}\ddot{q} \approx \Bigg[& M_h^{-1}\Bigg(-\frac{\partial v_h}{\partial q} - \frac{\partial J^{\mathrm{T}} \mathrm{t}_{\mathrm{eq}}}{\partial q} \\ & - J^{\mathrm{T}}\Bigg(KJ + \frac{\partial K\xi_{\mathrm{eq}}}{q}\Bigg)\Bigg)\Bigg]_{\mathrm{eq}} \delta q \\ & + \Bigg[M_h^{-1} J^{\mathrm{T}}\Bigg(KG^{\mathrm{T}} - \frac{\partial K\xi_{\mathrm{eq}}}{u}\Bigg)\Bigg]_{\mathrm{eq}} \delta u \\ & + [-M_h^{-1} J^{\mathrm{T}} BJ]_{\mathrm{eq}} \dot{q} + [M_h^{-1} J^{\mathrm{T}} BG^{\mathrm{T}}]_{\mathrm{eq}} \dot{u} \\ & + [M_h^{-1}]_{\mathrm{eq}} \delta\tau. \end{aligned} \tag{41}$$

Linearization of object dynamics (6) proceeds along similar lines. Putting

$$Q_o(u, \dot{u}) = C_O(u, \dot{u})\dot{u} + v_O(u)$$

and noting that $C_O(u, 0) = 0$ and that $(\partial v_O/u) = 0$, the first-order approximation of (6) is

$$\begin{aligned}\ddot{u} \approx & \Bigg[M_O^{-1} G\Bigg(KJ + \frac{\partial K\xi_{\mathrm{eq}}}{q}\Bigg)\Bigg]_{\mathrm{eq}} \delta q \\ & + \Bigg[M_O^{-1}\Bigg(\frac{\partial v_O}{\partial u} + \frac{\partial G \mathrm{t}_{\mathrm{eq}}}{\partial u} - G\Bigg(KG^{\mathrm{T}} - \frac{\partial K\xi_{\mathrm{eq}}}{u}\Bigg)\Bigg)\Bigg]_{\mathrm{eq}} \\ & \cdot \delta u + [M_O^{-1} GBJ]_{\mathrm{eq}} \dot{q} + [-M_O^{-1} GBG^{\mathrm{T}}]_{\mathrm{eq}} \dot{u} \\ & + [M_O^{-1}]_{\mathrm{eq}} \delta w. \end{aligned} \tag{42}$$

Rewriting (41) and (42) in the state-space description (8)–(10), one gets (43), shown at the bottom of the page, where v_h (v_O) denotes the gravitational parts of Q_h (Q_O). All the matrices of the linearized dynamic model are implicitly assumed to be evaluated at the equilibrium configuration.

In the general case, block L_k still has a rather involved expression in terms of the system's kinematic parameters and material properties, and depends on the intensity of forces at equilibrium. To the purpose of obtaining clearly intelligible results relating structural properties of manipulation systems to their more intrinsic parameters, the linearized model is considered under further assumptions as follows.

A3: Terms due to gravity v_h and v_O are null.

A4: Stiffness and damping are isotropic at each contact, i.e., there exists positive constants κ_i and β_i such that, in a local frame, $K_i = \kappa_i I$ and $B_i = \beta_i I$, with $\beta_i/\kappa_i =$ const. This implies that, in base frame, $(\partial K/\partial(q, u)) = (\partial B/\partial(q, u)) = 0$ and that $B \propto K$. This assumption is customary in mechanical vibration analysis [20].

A5: $J(q)$ and $G(u)$ are slowly varying functions of their arguments, so that terms $(\partial J^{\mathrm{T}} \mathrm{t}_{\mathrm{eq}}/\partial q), (\partial G \mathrm{t}_{\mathrm{eq}}/\partial u)$ are negligible. Note that assuming small contact forces at the equilibrium has the same effect on the linearizing approximation.

Accordingly, the simplified linearized dynamics (10)–(12) is obtained.

APPENDIX B

In order to prove Theorems 1 and 2, some preparatory lemmas are needed. For the sake of simplicity, we assume in the following that the representation of states is normalized so as to have homogeneous physical dimensions for all states. Under this assumption, it is possible to define an internal product in the state space $x^{\mathrm{T}} x$: $\mathbb{R}^{2(q+d)} \times \mathbb{R}^{2(q+d)} \to \mathbb{R}$), and the notion of orthogonality between subspaces. Since such a normalization of the state space can always be obtained by means of a linear transformation of coordinates that is positive definite, no loss of generality will ensue fom this assumption.

Lemma 2: The subspace unobservable from object displacements is controllable from joint torques, i.e., $\ker O_u \subseteq \langle A, B_\tau \rangle$.

Proof: Directly from comparison of (23) and (18). ■

Lemma 3: The subspace unobservable from joint displacements is controllable from external wrenches, i.e., $\ker O_q \subseteq \langle A, B_w \rangle$.

Proof: Directly from comparison of to (31) and (33). ■

Lemma 4: Inertia matrix M maps the subspace controllable from joint torques in the orthogonal complement to the subspace unobservable from joint displacements, i.e., $M\langle A, B_\tau \rangle = \ker O_q^{\perp}$. Moreover, $\langle A, B_\tau \rangle \oplus \ker O_q = \mathbb{R}^{2(q+d)}$.

Proof: By comparing (18) and (31), the thesis is proved by showing that

$$\begin{aligned}\mathcal{W}_h^{\perp} &= \left(\bigcap_{i=1}^{d} \ker\,[J^{\mathrm{T}} KG^{\mathrm{T}} (M_O^{-1} GKG^{\mathrm{T}})^{i-1}]\right)^{\perp} \\ &= \sum_{k=1}^{d} \mathrm{im}\,((GKG^{\mathrm{T}} M_O^{-1})^{k-1} GKJ) \\ &= M_O \sum_{k=1}^{d} \mathrm{im}\,((M_O^{-1} GKG^{\mathrm{T}})^{k-1} M_O^{-1} GKJ) \\ &= M_O \langle M_O^{-1} GKG^{\mathrm{T}}, M_O^{-1} GKJ \rangle. \end{aligned}$$

The rest of the proof follows from inertia matrix M being positive definite. ■

Lemma 5: Inertia matrix M maps the subspace controllable from external wrenches in the orthogonal complement to

$$\begin{aligned} L_k &= -\begin{bmatrix} M_h^{-1}\left(-\dfrac{\partial(v_h - J^{\mathrm{T}} \mathrm{t}_{\mathrm{eq}})}{\partial q} - J^{\mathrm{T}}\left(KJ + \dfrac{\partial K\xi_{\mathrm{eq}}}{\partial q}\right)\right) & M_h^{-1} J^{\mathrm{T}}\left(KG^{\mathrm{T}} - \dfrac{\partial K\xi_{\mathrm{eq}}}{u}\right) \\ M_O^{-1} G\left(KJ + \dfrac{\partial K\xi_{\mathrm{eq}}}{q}\right) & M_O^{-1}\left(\dfrac{\partial(v_O + G\mathrm{t}_{\mathrm{eq}})}{\partial u} - G\left(KG^{\mathrm{T}} - \dfrac{\partial K\xi_{\mathrm{eq}}}{u}\right)\right) \end{bmatrix} \\ L_b &= -\begin{bmatrix} -M_h^{-1} J^{\mathrm{T}} BJ & M_h^{-1} J^{\mathrm{T}} BG^{\mathrm{T}} \\ M_O^{-1} GBJ & -M_O^{-1} GBG^{\mathrm{T}} \end{bmatrix} \end{aligned} \tag{43}$$

the subspace unobservable from object displacements, i.e., $\boldsymbol{M}\langle\boldsymbol{A},\boldsymbol{B_w}\rangle = \ker\boldsymbol{O_u}^{\perp}$. Moreover, $\langle\boldsymbol{A},\boldsymbol{B_w}\rangle \oplus \ker\boldsymbol{O_u} = \mathbb{R}^{2(\boldsymbol{q}+\boldsymbol{d})}$.

Proof: Similar to proof of Lemma 4, with reference to (33) and (23). ■

Lemma 6: The inverse of inertia matrix $\boldsymbol{M}$ maps the orthogonal complement to the sum of the unobservable subspaces from object and joint displacements in the $\boldsymbol{A}$-invariant subspace of states controllable from both input torques and external wrenches, i.e., $\langle\boldsymbol{A},\boldsymbol{B}_\tau\rangle \cap \langle\boldsymbol{A},\boldsymbol{B_w}\rangle = \boldsymbol{M}^{-1}(\ker\boldsymbol{O_u}\oplus\ker\boldsymbol{O_q})^{\perp}$. Moreover, $(\langle\boldsymbol{A},\boldsymbol{B}_\tau\rangle\cap\langle\boldsymbol{A},\boldsymbol{B_w}\rangle)\oplus(\ker\boldsymbol{O_u}\oplus\ker\boldsymbol{O_q}) = \mathbb{R}^{2(q+d)}$.

Proof: From Lemmas 4 and 5

$$\begin{aligned}\boldsymbol{M}^{-1}&(\ker\boldsymbol{O_u}\oplus\ker\boldsymbol{O_q})^{\perp}\\ &= (\boldsymbol{M}^{-1}\ker\boldsymbol{O_u}^{\perp})\cap(\boldsymbol{M}^{-1}\ker\boldsymbol{O_q}^{\perp})\\ &= \langle\boldsymbol{A},\boldsymbol{B}_\tau\rangle\cap\langle\boldsymbol{A},\boldsymbol{B_w}\rangle.\end{aligned}$$

The rest of the proof follows from matrix $\boldsymbol{M}^{-1}$ being positive definite. ■

We prove now two previously anticipated results:

Lemma 7: The subspace unobservable from object displacements can be decomposed in the direct sum of two $\boldsymbol{A}$-invariant subspaces, one of which is the redundant subspace $\mathcal{X}_r$ (25). Equivalently, there exists a matrix $\mathrm{T}_{\boldsymbol{h}}$ such that

$$\begin{cases}\ker\boldsymbol{O_u} = \operatorname{im}\mathrm{T}_{\boldsymbol{r}} + \operatorname{im}\mathrm{T}_{\boldsymbol{h}}\\ \operatorname{im}\mathrm{T}_{\boldsymbol{r}}\cap\operatorname{im}\mathrm{T}_{\boldsymbol{h}} = \boldsymbol{0}\\ \operatorname{im}\boldsymbol{A}\mathrm{T}_{\boldsymbol{h}}\subseteq\operatorname{im}\mathrm{T}_{\boldsymbol{h}}.\end{cases}$$

Proof: Remind from (28) and (23) that

$$\mathrm{T}_{\boldsymbol{r}} = \begin{bmatrix}\Gamma_r & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \Gamma_r\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix};\quad \ker\boldsymbol{O_u} = \operatorname{im}\begin{bmatrix}\boldsymbol{V_h} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{V_h}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$$

where Γ_r is a b.m. of $\ker\boldsymbol{J}$, and $\boldsymbol{V_h}$ is a b.m. of $\mathcal{V}_h$ (24). The matrix $\mathrm{T}_{\boldsymbol{h}}$ satisfying the lemma is

$$\begin{aligned}&\mathrm{T}_{\boldsymbol{h}} = \begin{bmatrix}\Gamma_h & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \Gamma_h\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\quad\text{with}\\ &\Gamma_h = \text{b.m. of } im\,(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}})\cap\mathcal{V}_{\boldsymbol{h}}.\end{aligned}\tag{44}$$

The first two claims can be shown by restricting to the space of joint displacements $\mathbb{R}^q$ as

$$\begin{cases}\ker\boldsymbol{J}\cap\operatorname{im}\Gamma_h = \boldsymbol{0};\\ \mathcal{V}_h = \operatorname{im}\Gamma_r + \operatorname{im}\Gamma_h.\end{cases}$$

In fact, since $\boldsymbol{M_h}$ is p.d., $\operatorname{im}(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}})\cap\ker\boldsymbol{J} = \boldsymbol{0}$, and $(\operatorname{im}(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}})\cap\mathcal{V}_{\boldsymbol{h}}) + \ker\boldsymbol{J} = \mathcal{V}_{\boldsymbol{h}}$. The rest of the proof follows from observing that

$$\operatorname{im}(\boldsymbol{A}\mathrm{T}_{\boldsymbol{h}}) = \operatorname{im}\begin{bmatrix}\boldsymbol{0} & \Gamma_h\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{J}\Gamma_h & \boldsymbol{0}\\ \boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{J}\Gamma_h & \boldsymbol{0}\end{bmatrix}\subseteq\operatorname{im}\mathrm{T}_{\boldsymbol{h}}.$$

In fact, by definition of $\Gamma_h, \boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{J}\Gamma_{\boldsymbol{h}} = \boldsymbol{0}$, and $\operatorname{im}(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{J}\Gamma_{\boldsymbol{h}})\subseteq\operatorname{im}(\boldsymbol{M}_{\boldsymbol{h}}^{-1}\boldsymbol{J}^{\mathrm{T}})$, hence the $\boldsymbol{A}$-invariance of $\operatorname{im}\mathrm{T}_{\boldsymbol{h}}$ is proved. ■

Lemma 8: The subspace unobservable from joint displacements can be decomposed in the direct sum of two $\boldsymbol{A}$-invariant subspaces, one of which is the indeterminate subspace $\mathcal{X}_i$. Equivalently, there exist a matrix $\mathrm{T}_{\boldsymbol{d}}$ such that

$$\begin{aligned}\ker\boldsymbol{O_q} &= \operatorname{im}\mathrm{T}_{\boldsymbol{i}} + \operatorname{im}\mathrm{T}_{\boldsymbol{d}}\\ \operatorname{im}\mathrm{T}_{\boldsymbol{i}}\cap\operatorname{im}\mathrm{T}_{\boldsymbol{d}} &= \boldsymbol{0}\\ \operatorname{im}\boldsymbol{A}\mathrm{T}_{\boldsymbol{d}} &\subseteq\operatorname{im}\mathrm{T}_{\boldsymbol{d}}\end{aligned}.$$

Proof: Defining $\mathrm{T}_{\boldsymbol{d}}$ as

$$\begin{aligned}&\mathrm{T}_{\boldsymbol{d}} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \Gamma_d & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \Gamma_d\end{bmatrix}\quad\text{with}\\ &\Gamma_d = \text{b.m. of }\operatorname{im}(\boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G})\cap\mathcal{W}_{\boldsymbol{h}}\end{aligned}\tag{45}$$

the proof is similar to that of Lemma 7. ■

Proof: (**Theorem 1**) Lemmas 7 and 8 prove the structures of $\boldsymbol{C_q}\mathrm{T}$ and $\boldsymbol{C_u}\mathrm{T}$ and the $\boldsymbol{A}$-invariance of the column spaces of block matrices $\mathrm{T}_{\boldsymbol{r}}, \mathrm{T}_{\boldsymbol{h}}, \mathrm{T}_{\boldsymbol{i}}$, and $\mathrm{T}_{\boldsymbol{d}}$. The $\boldsymbol{A}$-invariance of $\operatorname{im}\mathrm{T}_{\boldsymbol{o}}$ follows directly from its definition: it is the intersection of $\boldsymbol{A}$-invariant subspaces. From Lemma 6, T is invertible, hence a valid change of coordinates. The definition of $\mathrm{T}_{\boldsymbol{o}}$ together with Lemmas 2 and 3 prove the structure of input and disturbance matrices. The proof ends by observing, directly from (30), the presence of two zero block matrices in $\boldsymbol{C}_{\mathrm{t}}\mathrm{T}$. ■

The following lemmas are instrumental to prove Theorem 2:

Lemma 9: A necessary and sufficient condition for the coordinated rigid motion subspace to be controllable from joint torques is that the image of its restriction to object displacements under the object inertia matrix is contained in the image of the grasp matrix, i.e.,

$$\operatorname{im}\mathrm{T}_{\boldsymbol{c}}\subseteq\langle\boldsymbol{A},\boldsymbol{B}_\tau\rangle\Leftrightarrow\boldsymbol{M_o}\operatorname{im}\Gamma_{uc}\subseteq\operatorname{im}\boldsymbol{G}.$$

Proof: Necessity is straightforward from (18) and from the fact that $\boldsymbol{M_o}\operatorname{im}\Gamma_{uc}\subseteq\boldsymbol{M_o}<\boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{G}^{\mathrm{T}},\boldsymbol{M}_{\boldsymbol{o}}^{-1}\boldsymbol{G}\boldsymbol{K}\boldsymbol{J}\rangle\subseteq$

$\mathrm{im}\,\boldsymbol{G}$. The sufficient part follows from the definition of Γ_{uc} [see (27)] and the Cayley–Hamilton theorem. ■

Lemma 10: A necessary and sufficient condition for the coordinated rigid motion subspace to be controllable from external wrenches is that the image of its restriction to joint displacements under the manipulator inertia matrix is contained in the image of the manipulator transpose Jacobian, i.e.,

$$\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}} \subseteq \langle \boldsymbol{A}, \boldsymbol{B_w} \rangle \Leftrightarrow \boldsymbol{M_h}\,\mathrm{im}\,\Gamma_{qc} \subseteq \mathrm{im}\,\boldsymbol{J}^{\mathrm{T}}.$$

Proof: Similar to proof of Lemma 9. ■

Proof: (**Theorem 2**) Lemmas 9 and 10 prove that condition (34) is necessary and sufficient for $\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}} \subseteq \mathrm{im}\,\mathrm{T}_{\boldsymbol{o}}$. Thus according to Theorem 1 and reminding that $\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}}$ is not observable from contact forces (29), it remains to show the $\boldsymbol{A}$-invariance of $\mathrm{im}\,\mathrm{T}_{\boldsymbol{a}}$. From the structure of matrix $\boldsymbol{A}$ (10) and the following equivalence:

$$\boldsymbol{A}\boldsymbol{M}^{-1}\,\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}}^{\perp} \subseteq \boldsymbol{M}^{-1}\,\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}}^{\perp} \Leftrightarrow \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\,\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}} \subseteq \boldsymbol{M}\,\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}}$$

it is an easy matter to verify the $\boldsymbol{A}$-invariance of $\boldsymbol{M}^{-1}\,\mathrm{im}\,\mathrm{T}_{\boldsymbol{c}}^{\perp}$. Thus, being the column space of $\mathrm{T}_{\boldsymbol{o}}$ $\boldsymbol{A}$-invariant as well, the invariance of $\mathrm{im}\,\mathrm{T}_{\boldsymbol{a}}$ is shown and the proof ends. ■

APPENDIX C

In this appendix numerical results are reported for the examples of Fig. 1. All manipulators are planar, and the manipulated object is assumed to be a disk of unit radius, mass, and barycentral moment of inertia. Moreover links have unit mass which is concentrated at their tips, so that barycentral moments of inertia are zero. Links are assumed to have unit length if there is no contacts with the object otherwise the distance between the contact point and the nearest joint is unitary. Stiffness and damping matrices at every contact are assumed to be normalized to the identity matrix.

Example 1: The manipulation system in Fig. 9 is nonredundant, determinate, graspable, defective, and nonhyperstatic

$$\boldsymbol{J} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \boldsymbol{G}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{T} = \mathrm{im} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \ker \boldsymbol{G} = \mathrm{im} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix};$$

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} = \boldsymbol{0}$$

$$\ker \boldsymbol{J} = \boldsymbol{0}; \quad \ker \boldsymbol{G}^{T} = \boldsymbol{0}$$

$$\mathrm{T} = [\mathrm{T}_a \quad \mathrm{T}_d]; \quad \mathrm{T}^{-1}\boldsymbol{A}\mathrm{T} = \left[\begin{array}{c|c} {}^{a}\boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & {}^{d}\boldsymbol{A} \end{array}\right]$$

$$\dim(\mathrm{im}\,\mathrm{T}_{\boldsymbol{a}}) = 4; \quad \dim(\mathrm{im}\,\mathrm{T}_{\boldsymbol{d}}) = 4.$$

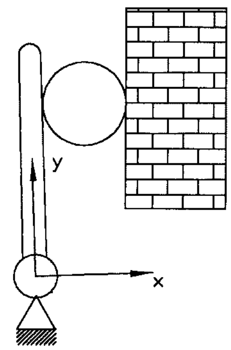

Fig. 9. Manipulation system of example 1.

Example 2: The manipulation system in Fig. 10 is nonredundant, indeterminate, nongraspable, defective, and nonhyperstatic

$$\boldsymbol{J} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}; \quad \boldsymbol{G}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} = \mathrm{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \ker \boldsymbol{G} = \boldsymbol{0}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} = \boldsymbol{0}$$

$$\ker \boldsymbol{J} = \boldsymbol{0}; \quad \ker \boldsymbol{G}^{\mathrm{T}} = \mathrm{im} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\mathrm{T} = [\mathrm{T}_i \quad \mathrm{T}_o]; \quad \mathrm{T}^{-1}\boldsymbol{A}\mathrm{T} = \left[\begin{array}{c|c} {}^{i}\boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & {}^{o}\boldsymbol{A} \end{array}\right]$$

$$\dim(\mathrm{im}\,\mathrm{T}_i) = 2; \quad \dim(\mathrm{im}\,\mathrm{T}_o) = 6.$$

Example 3: The manipulation system in Fig. 11 is nonredundant, determinate, graspable, defective, and nonhyperstatic

$$\boldsymbol{J} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}; \quad \boldsymbol{G}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} = \mathrm{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad \ker \boldsymbol{G} = \mathrm{im} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} = \boldsymbol{0}$$

$$\ker \boldsymbol{J} = \boldsymbol{0}; \quad \ker \boldsymbol{G}^{\mathrm{T}} = \boldsymbol{0}$$

$$\hat{\mathrm{T}} = [\mathrm{T}_h \quad \mathrm{T}_d \quad \mathrm{T}_c \quad \mathrm{T}_a]$$

$$\hat{\mathrm{T}}^{-1}\boldsymbol{A}\hat{\mathrm{T}} = \left[\begin{array}{c|c|c|c} {}^{h}\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & {}^{d}\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & {}^{c}\boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & {}^{a}\boldsymbol{A} \end{array}\right]$$

$$\dim(\mathrm{im}\,\mathrm{T}_h) = 2; \quad \dim(\mathrm{im}\,\mathrm{T}_d) = 4;$$

$$\dim(\mathrm{im}\,\mathrm{T}_c) = 2; \quad \dim(\mathrm{im}\,\mathrm{T}_a) = 2.$$

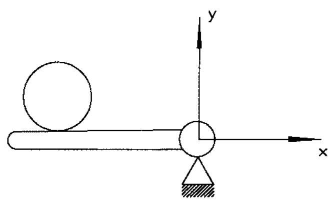

Fig. 10. Manipulation system of example 2.

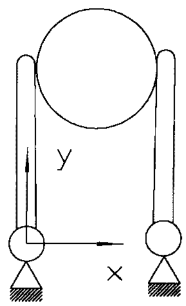

Fig. 11. Manipulation system of example 3.

Example 4:

The manipulation system in Fig. 12 is nonredundant, determinate, graspable, defective, and hyperstatic

$$\boldsymbol{J} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\boldsymbol{G}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} = \mathrm{im} \begin{bmatrix} 0.3 & -0.6 & 0.59 \\ -0.3 & -0.7 & -0.6 \\ -0.1 & 0.3 & -0.3 \\ 0.8 & 0 & -0.4 \\ 0 & 0 & 0 \\ -0.4 & 0 & 0.2 \end{bmatrix}$$

$$\ker \boldsymbol{G} = \mathrm{im} \begin{bmatrix} -0.1 & -0.7 & -0.4 \\ -0.5 & 0.1 & -0.2 \\ 0.2 & -0.1 & 0.7 \\ 0.8 & 0 & -0.2 \\ -0.1 & 0.8 & -0.2 \\ -0.3 & -0.1 & 0.4 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} = \mathrm{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker \boldsymbol{J} = \boldsymbol{0}; \quad \ker \boldsymbol{G}^{\mathrm{T}} = \boldsymbol{0}$$

$$\hat{\mathrm{T}} = [\mathrm{T}_c \quad \mathrm{T}_a]; \quad \hat{\mathrm{T}}^{-1} \boldsymbol{A} \hat{\mathrm{T}} = \left[\begin{array}{c|c} {}^c\boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & {}^a\boldsymbol{A} \end{array}\right]$$

$$\dim(\mathrm{im}\,\mathrm{T}_c) = 2; \quad \dim(\mathrm{im}\,\mathrm{T}_a) = 10.$$

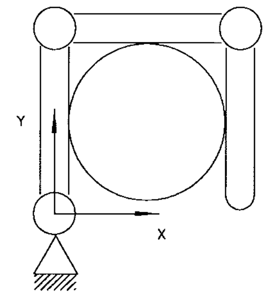

Fig. 12. Manipulation system of example 4.

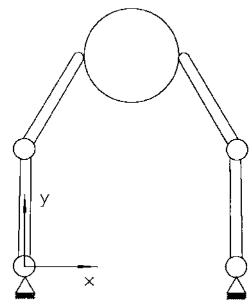

Fig. 13. Manipulation system of example 5.

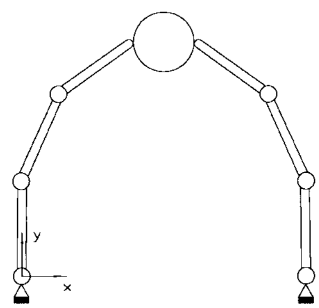

Fig. 14. Manipulation system of example 6.

Example 5: The manipulation system in Fig. 13 is nonredundant, determinate, graspable, nondefective, and nonhyperstatic

$$\boldsymbol{J} = \begin{bmatrix} -1.9 & -0.87 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & -1.9 & -0.87 \\ 0 & 0 & -0.5 & -0.5 \end{bmatrix}$$

$$\boldsymbol{G}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} = \boldsymbol{0}; \quad \ker \boldsymbol{G} = \mathrm{im} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} = \boldsymbol{0}$$

$$\ker \boldsymbol{J} = \boldsymbol{0}; \quad \ker \boldsymbol{G}^{\mathrm{T}} = \boldsymbol{0}$$

$$\hat{\mathrm{T}} = [\mathrm{T}_c \quad \mathrm{T}_a]; \quad \hat{\mathrm{T}}^{-1} \boldsymbol{A} \hat{\mathrm{T}} = \left[\begin{array}{c|c} {}^c\boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & {}^a\boldsymbol{A} \end{array}\right]$$

$$\dim(\mathrm{im}\,\mathrm{T}_c) = 6; \quad \dim(\mathrm{im}\,\mathrm{T}_a) = 8.$$

Example 6: The manipulation system in Fig. 14 is redundant, determinate, graspable, nondefective, and nonhyperstatic

$$\boldsymbol{J} = \begin{bmatrix} -2.7 & -1.7 & -0.87 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.7 & -1.7 & -0.87 \\ 0 & 0 & 0 & -1 & -1 & -0.5 \end{bmatrix}$$

$$\boldsymbol{G}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} = \mathbf{0}; \quad \ker \boldsymbol{G} = \operatorname{im} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\ker \boldsymbol{J}^{\mathrm{T}} \cap \ker \boldsymbol{G} = \mathbf{0}$$

$$\ker \boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 2 \end{bmatrix} \quad \ker \boldsymbol{G}^{\mathrm{T}} = \mathbf{0}$$

$$\hat{\mathrm{T}} = [\mathrm{T}_r \quad \mathrm{T}_c \quad \mathrm{T}_a]$$

$$\hat{\mathrm{T}}^{-1}\boldsymbol{A}\hat{\mathrm{T}} = \left[\begin{array}{c|c|c} {}^r\boldsymbol{A} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & {}^c\boldsymbol{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & {}^a\boldsymbol{A} \end{array}\right]$$

$$\dim(\operatorname{im} \mathrm{T}_r) = 4; \quad \dim(\operatorname{im} \mathrm{T}_c) = 6; \quad \dim(\operatorname{im} \mathrm{T}_a) = 8.$$

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